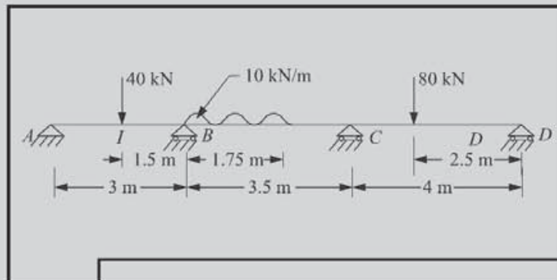
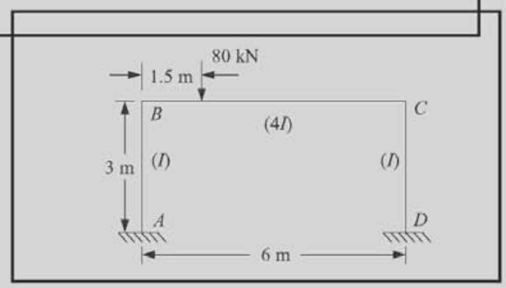
Chapter 3

Kani's Method of Rotation Contribution





Chapter Outline

3.1 INTRODUCTION

Gasper Kani, a German engineer, developed another distribution procedure based on slope deflection equations. This method is very useful for the analysis of multistorey frames. The greatest advantage of this method is, even if a mistake is committed in distribution in one of the cycles, it converges finally to the correct answer. Even today, many practising engineers who are not familiar with computer methods, use Kani's method for the analysis of 3 to 4 storey building frames.

This method is first explained for structures with fixed ends. Then, the modifications to handle simply supported and overhanging ends are discussed. Analysis of symmetric frames making use of the symmetry is also explained, after which an analysis of general frames is taken up.

Sign Conventions In above method, the following sign conventions are used:

1. Clockwise end moments are positive; 2. Clockwise rotations are positive

3.2 ANALYSIS OF STRUCTURES WITHOUT RELATIVE DISPLACEMENT AT ENDS

Member *AB*, shown in Figure 3.1(*a*), is an intermediate member of a beam/frame, which has no relative displacements at the ends (*i.e.*, ends *A* and *B* are at the same level).

Let M_{AB} and M_{BA} be the final end moments. M_{AB} may consist of:

(i) Fixed end moments ($\theta_A = \theta_B = 0$) (Figure 3.1(*b*))
(ii) Moment due to rotation of end *A* only (Figure 3.1(*c*))
(iii) Moment due to rotation of end *B* only (Figure 3.1(*d*))

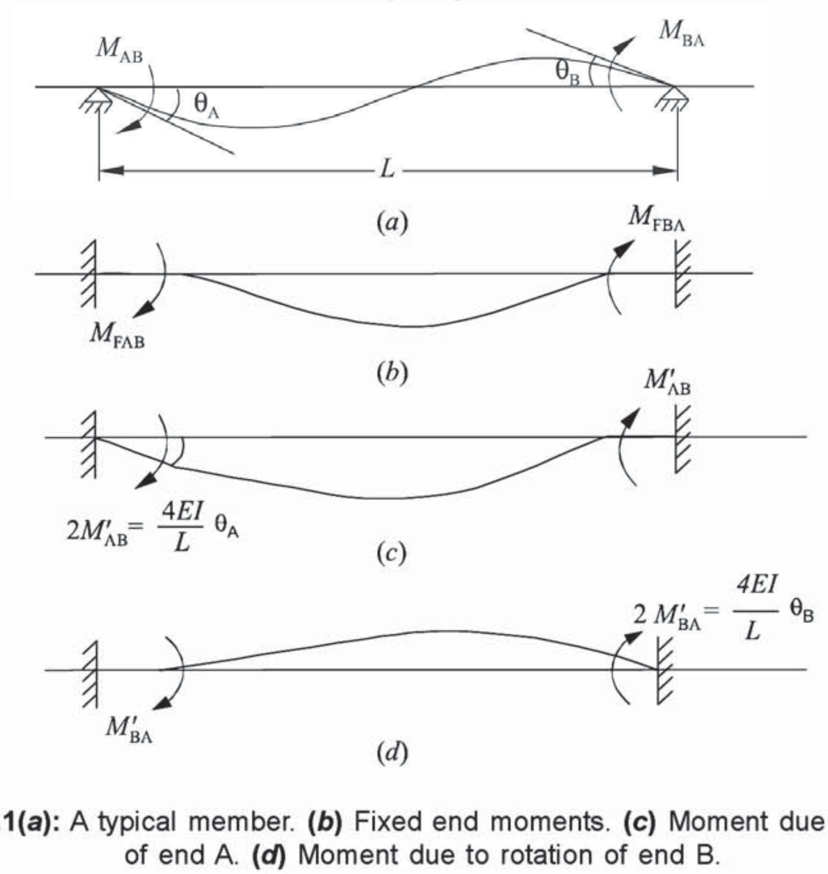


Figure 3.1(*a*): A typical member. **(*b*)** Fixed end moments. **(*c*)** Moment due to rotation of end A. **(*d*)** Moment due to rotation of end B.

> **Note:** *In Figure 3.1, all the moments are shown in their positive signs. Some of them may have negative values.*

Let the moment developed at A due to rotation θ_A only be $2M'_{AB}$. Naturally, it is equal to $\frac{4EI}{L}\theta_A$.

Hence, moment developed at $B = M'_{AB}$.

Similarly, the moments developed at ends A and B due to rotation θ_B only are M'_{BA} and $2M'_{BA} = \left(\frac{4EI}{L}\theta_B\right)$ respectively.

$$\therefore \quad M_{AB} = M_{FAB} + 2M'_{AB} + M'_{BA} \quad \ldots(3.1a)$$

and

$$M_{BA} = M_{FBA} + M'_{AB} + 2M'_{BA} \quad \ldots(3.1b)$$

The moments M'_{AB} and M'_{BA} are called *rotation contributions*. In general, Eqn. 3.1 may be stated as

Final moment = Fixed end moment + 2 (Rotation contribution of near end)
+ Rotation contribution of far end ...(3.2)

Now, consider the moments at joint A in the frame shown in Figure 3.2.

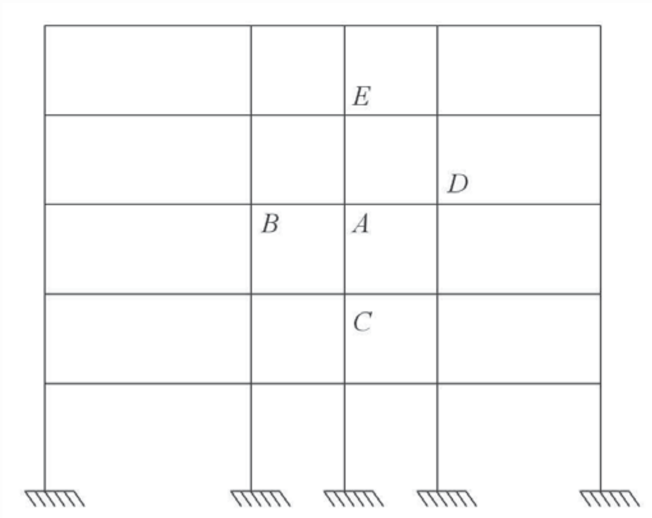


Figure 3.2: A typical frame.

According to Eqn. (3.2),

$$M_{AB} = M_{FAB} + 2M'_{AB} + M'_{BA}$$
$$M_{AC} = M_{FAC} + 2M'_{AC} + M'_{CA}$$
$$M_{AD} = M_{FAD} + 2M'_{AD} + M'_{DA}$$

and

$$M_{AE} = M_{FAE} + 2M'_{AE} + M'_{EA}$$
$$\Sigma M_{AB} = \Sigma M_{FAB} + 2\Sigma M'_{AB} + \Sigma M'_{BA}$$

where ΣM_{AB} = sum of near end moments in all the members meeting at joint A

$\Sigma M'_{FAB}$ = sum of fixed end moments in all the members at joint A

$\Sigma M'_{AB}$ = sum of near end rotation contributions of all the members meeting at joint A

$\Sigma M'_{BA}$ = sum of far end rotation contributions of all the members meeting at joint A

From the joint equilibrium condition, we know,

$$\Sigma M_{AB} = 0$$

$$\therefore \qquad \Sigma M_{FAB} + 2\,\Sigma M'_{AB} + \Sigma M'_{BA} = 0$$

$$\Sigma M'_{AB} = -\frac{1}{2}(\Sigma M_{FAB} + \Sigma M'_{BA}) \qquad \ldots(a)$$

For each member meeting at joint A,

$$2\,M'_{AB} = \left(\frac{4\,EI}{L}\right)\theta_A = k_{AB}\theta_A$$

$$\therefore \qquad M'_{AB} = \frac{1}{2}k_{AB}\,\theta_A$$

Hence,

$$\Sigma M'_{AB} = \frac{1}{2}\Sigma k_{AB}\,\theta_A$$

$$= \frac{1}{2}\theta_A\;\Sigma k_{AB},$$

Since, θ_A is the same for all the members meeting at joint A.

$$\therefore \qquad \frac{M'_{AB}}{\Sigma M'_{AB}} = \frac{k_{AB}}{\Sigma k_{AB}}$$

$$M'_{AB} = \left(\frac{k_{AB}}{\Sigma k_{AB}}\right)\Sigma M'_{AB} \qquad \ldots(b)$$

Substituting Eqn. (a) in Eqn. (b), we get,

$$M'_{AB} = -\frac{1}{2}\left(\frac{k_{AB}}{\Sigma k_{AB}}\right)(\Sigma M_{AB} + \Sigma M'_{AB}) \qquad \ldots(3.3)$$

The expression $-\frac{1}{2}\left(\frac{k_{AB}}{\Sigma k_{AB}}\right)$ is called the *Rotation Factor* (RF) for member AB at joint A. From this equation, Kani developed the rotation contribution method.

In any given problem, the fixed end moments at all joints can be found. Hence, at any joint, ΣM_{FAB}, can be found. To calculate the rotation contributions from Eqn. 3.3, we require the far end contributions which are not known. Assuming them to be zero, calculate the near end contribution using Eqn. (3.3). Likewise, we calculate the near end rotation contribution for the next joint taking the far end contribution, if available, or, otherwise, assuming it to be zero. In this manner, we calculate the rotation contributions at all joints, which completes the first cycle. Now, the rotation contributions at the far ends are also available. Using Eqn. 3.3 again, we calculate the near end contributions at all the joints to complete the second cycle. Repeat the procedure until a change in the rotation contributions of two successive iterations is negligible. Then, calculate the final moments using Eqn. 3.2.

3.2.1 Application of Kani's Method to Continuous Beams with Fixed Ends

The above procedure may be applied to continuous beams with fixed ends noting that the rotation contribution at the end is zero, since the end being fixed, its rotation is zero.

Example 3.1 Analyse the continuous beam shown in Figure 3.3 by Kani's method.

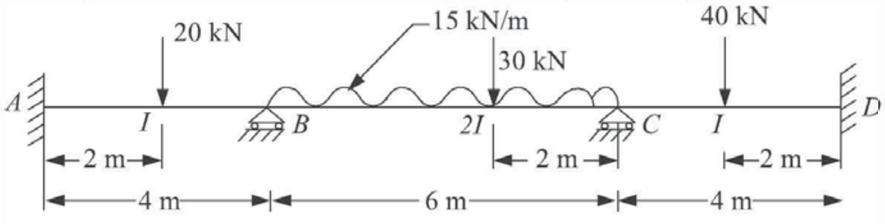


Figure 3.3: Continuous beam.

Fixed End Moments

$$M_{FAB} = -\frac{20 \times 4}{8} = -10 \text{ kNm}$$

$$M_{FBA} = 10 \text{ kNm}$$

$$M_{FBC} = -\frac{15 \times 6^2}{12} - \frac{30 \times 4 \times 2^2}{6^2} = -58.33 \text{ kNm}$$

$$M_{FCB} = \frac{15 \times 6^2}{12} + \frac{30 \times 4^2 \times 2}{6^2} = 71.67 \text{ kNm}$$

$$M_{FCD} = -\frac{40 \times 4}{8} = -20 \text{ kNm}$$

$$M_{FDC} = 20 \text{ kNm}$$

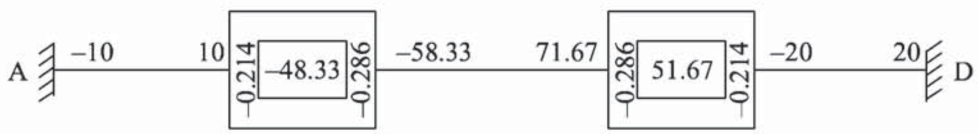
$$\text{Rotation factor (RF)} = -\frac{1}{2}\left(\frac{k_{AB}}{\Sigma k}\right)$$

Table 3.1: Rotation factors

Joint	*Members*	k	Σk	*RF*
B	*BA*	$\frac{4EI}{4} = EI$	2.33 *EI*	–0.214
	BC	$\frac{4E(2I)}{6} = \frac{4}{3}EI$		–0.286
C	*CB*	$\frac{4E(2I)}{6} = \frac{4}{3}EI$	2.33 *EI*	–0.286
	CD	$\frac{4EI}{4} = EI$		–0.214

Rotation contribution are calculated in the tabular form shown in Table 3.2.

Table 3.2: Rotation contributions



10.34	13.84	–18.73	–14.01
14.35	19.18	–20.26	–15.16
14.68	19.62	–20.39	–15.26
14.70	19.65	–20.40	–15.26

Explanation In Table 3.2, double rectangular blocks are made at joints *B* and *C* which rotate during deformation. End *A* and *D* are marked as the fixed ends. The fixed end moments –10.10; –58.33, 71.67; –20, 20 are marked above the horizontal line drawn to connect joints. ΣM_{FAB} terms are calculated and noted in the respective inner rectangular blocks of joints.

In this problem, for joint *B*,

$$\Sigma M_{FAB} = 10 - 58.33 = -48.33$$

and for joint *C*, $\Sigma M_{FAB} = 71.67 - 20 = 51.67$

Then, rotation factors – 0.214, – 0.286 at joint *B* and – 0.286 and – 0.214 at joint *C* are noted in the space between the two rectangular blocks as shown in Table 3.2.

Now, at joint *B*, rotation contribution terms M'_{BA} and M'_{BC} are to be found. From Eqn. 3.3,

M'_{BA} = *RF* (Σ Fixed end moments at joint *B* + Σ Rotation contributions from far end)

Rotation contribution for the far end of *BA* is zero, since, the far end *A* is fixed and not rotating. Rotation contribution at the far end of *BC* is not known. Hence, it is taken as zero.

$$M'_{BA} = -0.214\,[-48.33 + 0] = 10.34$$

$$M'_{BC} = -0.286\,[-48.33 + 0] = 13.82$$

These values are noted below the members near the joint.

Now, consider joint *C*. For this joint,

Σ Fixed end moments = 51.67, as shown in rectangular block. There is no far end contribution from member *CD*, since end *D* is fixed.

The far end contribution of *CB*, *i.e.*, from end *B*, is 13.82

∴ Σ Far end contribution = 13.82 + 0 = 13.82

∴ $$M'_{CB} = -0.286\ [51.67 + 13.82] = -18.73$$

and $$M'_{CD} = -0.214\ [51.67 + 13.82] = -14.01$$

All joints which rotate have been considered. Hence, a cycle is completed.

The second cycle again starts from joint *B*. The far end contribution of member *BC* is –18.73; there is no far end contribution for member *BA*.

∴ Σ Far end contribution = – 18.73

$$M'_{BA} = -0.214\ [-48.33 - 18.73] = 14.35$$

$\therefore$ $$M'_{BC} = -0.286\ [-48.33 - 18.73] = 19.18$$

Consider joint C,

$$M'_{CB} = -0.286\ [51.67 + 19.18] = -20.26$$

$$M'_{CD} = -0.214\ [51.67 + 19.18] = -15.16$$

Similarly, further cycles are carried out. The difference between rotation contributions of cycle 4 and cycle 3 are negligible. Hence, the distribution procedure is stopped.

From Eqn. 3.2, we know

Final moment = Fixed end moment + 2 (Near end rotation contribution) + Far end contribution

Table 3.3: Final moment calculations

	A		*B*		*C*	*D*
FEM	–10	10	–58.33	71.67	–20	20
Near end conribution	2 × 0	2 × 14.7	2 × 19.65	–2 × 20.40	2 × (–15.16)	2 × 0
Far end conribution	14:70	0	–20.40	19.65	0	–15.26
Final	4.70	39.4	–39.4	50.52	–50.52	4.74

Example 3.2 Analyse the continuous beam shown in Figure 3.4 by Kani's method.

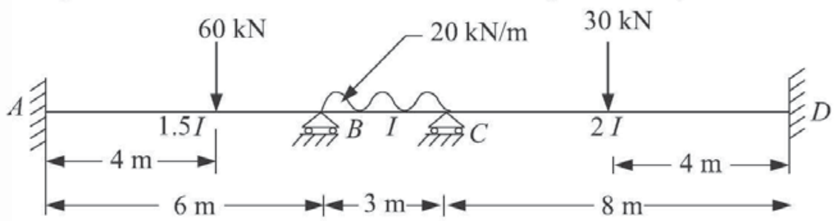


Figure 3.4: Continuous beam.

Solution ***Fixed End Moments***

$$M_{FAB} = -\frac{60 \times 4 \times 2^2}{6^2} = -26.67 \text{ kNm}$$

$$M_{FBA} = \frac{60 \times 4^2 \times 2}{6^2} = 53.33 \text{ kNm}$$

$$M_{FBC} = -\frac{20 \times 3^2}{12} = -15 \text{ kNm}$$

$$M_{FCB} = 15 \text{ kNm}$$

$$M_{FCD} = -\frac{30 \times 8}{8} = -30 \text{ kNm}$$

$$M_{FDC} = 30 \text{ kNm}$$

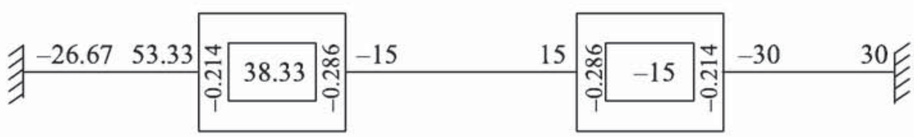
Rotation factor (RF) $= -\dfrac{1}{2}\left(\dfrac{k}{\Sigma k}\right)$

Table 3.4: Rotation factors

Joints	*Members*	*k*	Σ *k*	*RF*
B	*BA*	$\frac{4E(1.5I)}{6} = EI$	2.33 *EI*	−0.214
	BC	$\frac{4EI}{3}$		−0.286
C	*CB*	$\frac{4EI}{3}$	2.33 *EI*	−0.286
	CD	$\frac{4E(2I)}{8} = EI$		−0.214

Further calculations are carried out in Table 3.5.

Table 3.5: Rotation contributions



−8.20	−10.96	7.43	5.56
−9.79	−13.09	8.03	6.01
−9.92	−13.26	8.08	6.04
−9.93	−13.27	8.09	6.05

Final Moment Calculations

Table 3.6: Final moment calculations

	A	*B*		*C*		*D*
FEM	−26.67	53.33	−15	15	−30	30
Near end contribution	2 × 0	−2 × 9.93	−2 × 13.27	2 × 8.09	2 × 6.05	2 × 0
Far end conribution	−9.93	0	8.09	−13.27	0	−6.05
Final	−36.60	33.74	−33.74	17.91	−17.91	36.05

3.2.2 Application to Continuous Beams with Simply Supported and Overhanging Ends

Method I If the end of the continuous beam is simply supported or has overhang, the last support also rotates. Hence, rotation contribution of that joint also should be found. Stiffness of overhanging portion may be taken as zero, since the moment in this portion does not depend upon the loading in the other portion. Then, rotation factor for interior member at such joint is −0.5.

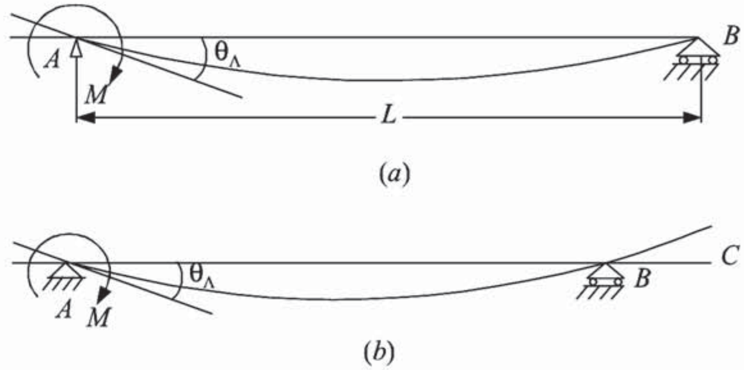


Figure 3.5: Rotation factors.

Method II The final moment at the simply supported end is zero and at the end support in the overhanging beam is equal to the end moment in the cantilever. Referring to Figure 3.5, we know that rotation at the inner end A due to moment M applied is

$$\theta_A = \frac{ML}{3\,EI}$$

$\therefore$ Moment required for unit rotation is $\frac{3\,EI}{L}$. Thus, stiffness of member AB is $\frac{3\,EI}{L}$.

If stiffness of member AB is modified as $\frac{3\,EI}{L}$, no moment gets transferred from end A to end B due to the rotation of end A. In the first step, fixed end moments are calculated at end B also, as if it is a fixed end. To achieve the final moment at end B (zero in case of simply supported end and at cantilever end moment in overhanging beam), balancing moment shall be applied at joint B. Then, half of this balancing moment goes to inner end A. Thus, fixed end moment of inner end A is modified as

$$\text{Modified} \quad M_{FAB} = M_{FAB} + 0.5 \times \text{Balancing moment at } B$$
$$= M_{FAB} - 0.5 \times \text{Unbalanced moment at } B$$

Method I is a direct application of Kani's concept of rotation contribution, while method II is a modification using the concept of moment distribution. In method II, the number of joints to be considered for calculating the rotation contribution is reduced. Hence, calculation effort is reduced.

Example 3.3 is solved by both the methods while the rest are solved by method II, only.

Example 3.3 Analyse the continuous beam shown in Figure 3.6 by Kani's method. Flexural rigidity is constant throughout.

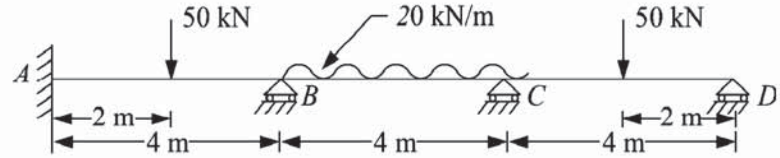


Figure 3.6: Continuous beam.

Solution ***Fixed End Moments***

$$M_{FAB} = -\frac{50 \times 4}{8} = -25 \text{ kNm}$$

$$M_{FBA} = 25 \text{ kNm}$$

$$M_{FBC} = -\frac{20 \times 4^2}{12} = -26.6 \text{ kNm}$$

$$M_{FCB} = 26.67 \text{ kNm}$$

$$M_{FCD} = -\frac{50 \times 4}{8} = -25 \text{ kNm}$$

$$M_{FDC} = 25 \text{ kNm}$$

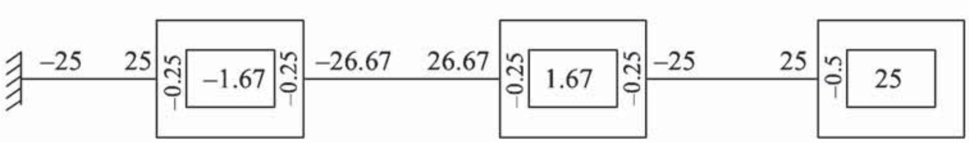
Method I Rotation factor (RF) $= -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$

Table 3.7: Rotation factors

Joints	*Members*	*k*	Σ *k*	*RF*
B	*BA*	$\frac{4EI}{4} = EI$	2 *EI*	–0.25
	BC	$\frac{4EI}{4} = EI$		–0.25
C	*CB*	$\frac{4EI}{4} = EI$	2 *EI*	–0.25
	CD	$\frac{4EI}{4} = EI$		–0.25
D	*DC*	$\frac{4EI}{4} = EI$	*EI*	–0.5

Rotation contributions and final moments are calculated and tabulated in Table 3.8.

Table 3.8: Rotation contributions



0.42	0.42	–0.52	–0.52	–12.24
0.55	0.55	2.51	2.51	–13.76
–0.21	–0.21	3.08	3.08	–14.04
–0.35	–0.35	3.18	3.18	–14.09
–0.38	–0.38	3.2	3.2	–14.10

Final Moment Calculations

Table 3.9: Final moment calculations

	A	*B*		*C*		*D*
FEM	–25	25	–26.67	26.67	–25	25
Near end contribution	2×0	$2 \times (-0.38)$	$2 \times (-0.38)$	2×3.2	2×3.2	$2 \times (-14.1)$
Far end conribution	–0.38	0	3.2	–0.38	–14.1	3.2
Final	–25.38	24.24	–24.23	32.69	–32.70	0

> **Note:** *While calculating the rotation contribution at C, the far end contributions of end B as well as end D are to be considered. Final moment expression used is Final moment = FEM + 2 (Near end rotation contribution) + (Far end contribution).*

Method II Fixed End Moments

Modification of FEM in the last span is required.

$$M_{FCD} = M_{FCD} - 0.5 \times \text{Unbalanced moment at } D$$
$$= -25 - 0.5(25) = -37.5 \text{ kNm}$$
$$M_{FDC} = 0$$

$$\text{Rotation factor (RF)} = -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

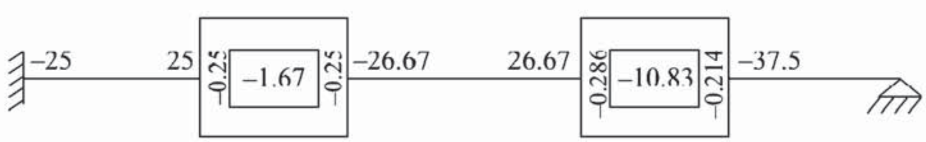
$$\text{Stiffness of span } CD \text{ is } \frac{3\,EI}{L} = \frac{3\,EI}{4}$$

Table 3.10: Rotation factors

Joint	*Members*	*k*	Σk	*RF*
B	*BA*	$\frac{4EI}{4} = EI$	2 *EI*	–0.25
	BC	$\frac{4EI}{4} = EI$		–0.25
C	*CB*	$\frac{4EI}{4} = EI$	1.75 *EI*	–0.286
	CD	$\frac{3EI}{4} = 0.75\,EI$		–0.214

Rotation and final moments are calculated and are shown in Table 3.11.

Table 3.11: Rotation contributions



0.42	0.42	2.98	2.23
–0.33	–0.33	3.19	2.39
–0.38	–0.38	3.21	2.39

Final Moment Calculations

Table 3.12: Final moment calculations

	A		*B*		*C*	*D*
FEM Near end contribution	–25 2 × 0	25 2 × (–0.38)	–26.67 2 × (–0.38)	26.67 2 × 3.21	–37.5 2 × 3.29	0 2 × 0
Far end contribution	–0.38	0	3.21	–0.38	0	0
Final	–25.38	24.24	–24.24	32.71	–32.71	0

Example 3.4 Analyse the continuous beam shown in Figure 3.7.

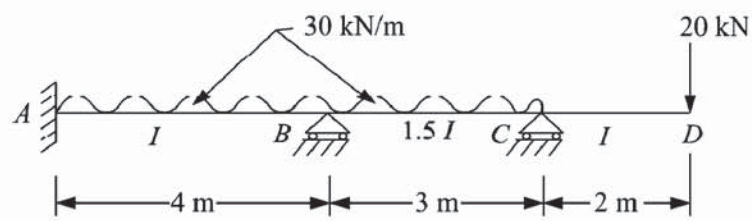


Figure 3.7: Continuous beam.

Solution ***Fixed End Moments***

$$M_{FAB} = -\frac{30 \times 4^2}{12} = -40 \text{ kNm}$$

$$M_{FBA} = 40 \text{ kNm}$$

$$M_{FBC} = -\frac{30 \times 3^2}{12} = -22.5 \text{ kNm}$$

$$M_{FCB} = 22.5 \text{ kNm}$$

$$M_{FCD} = -20 \times 2 = -40 \text{ kNm}$$

Modification to take care of rotation of support *C*

Modified $M_{FBC} = M_{FBC} - 0.5 \times$ Unbalanced moment at *C*

$$M_{FBC} = -22.5 - 0.5(22.5 - 40) = -13.5 \text{ kNm}$$

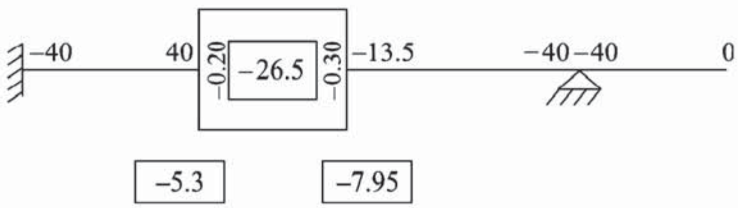
$$\text{Rotation factor (RF)} = -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.13: Rotation factors

Joint	*Members*	*k*	Σ *k*	*RF*
B	*BA*	$\frac{4EI}{4} = EI$	2.5 *EI*	–0.2
	BC	$\frac{3E(1.5I)}{3} = 1.5\,EI$		–0.3

Rotation contributions and final moments are calculated in Table 3.14.

Table 3.14: Rotation contributions



Final Moment Calculations

Table 3.15: Final moment calculations

	A	B		C		D
FEM	–40	40	– 13.5	40	–40	0
Near end contribution	2 × 0	2 (–5.3)	2 × (–7.95)	0	0	0
Far end contribution	–5.33	0	0	0	0	0
Final	– 45.33	29.40	–29.40	40	–40	0

Example 3.5 Analyse the continuous beam shown in Figure 3.8 by Kani's method. *EI* is constant throughout.

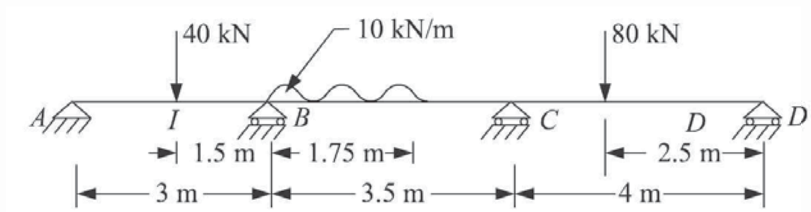


Figure 3.8: Continuous beam.

Solution ***Fixed End Moments***

$$M_{\text{FAB}} = -\frac{40 \times 3}{8} = -15 \text{ kNm}$$

$$M_{\text{FBA}} = 15 \text{ kNm}$$

$$M_{\text{FBC}} = -\int_0^{1.75} \frac{10\,dx \times x\,(3.5 - x)^2}{3.5^2}$$

$$= -\frac{10}{3.5^2}\int_0^{1.75} (3.5^2 x - 7x^2 + x^3)\,dx$$

$$= -\frac{10}{3.5^2}\left[3.5^2\left(\frac{x^2}{2}\right) - 7\left(\frac{x^3}{3}\right) + \frac{x^4}{4}\right]_0^{1.75} = -7.02 \text{ kNm}$$

$$M_{\text{FCB}} = \int_0^{1.75} \frac{10\,dx \times x^2\,(3.5 - x)}{3.5^2}$$

$$= \frac{10}{3.5^2}\int_0^{1.75}(3.5x^2 - x^3)\,dx$$

$$= \frac{10}{3.5^2}\left[3.5\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_0^{1.75} = 3.19 \text{ kNm}$$

$$M_{FCD} = \frac{-80 \times 1.5 \times 2.5^2}{4^2} = -46.88 \text{ kNm}$$

$$M_{FDC} = \frac{-80 \times 1.5^2 \times 2.5}{4^2} = 28.13$$

Modification to account for rotation of A and D

$$M_{FAB} = 0$$

$$M_{FBA} = 15 - 0.5(-15) = 22.5 \text{ kNm}$$

$$M_{FCD} = -46.88 - 0.5(28.13) = -60.95 \text{ kNm}$$

$$M_{FDC} = 0$$

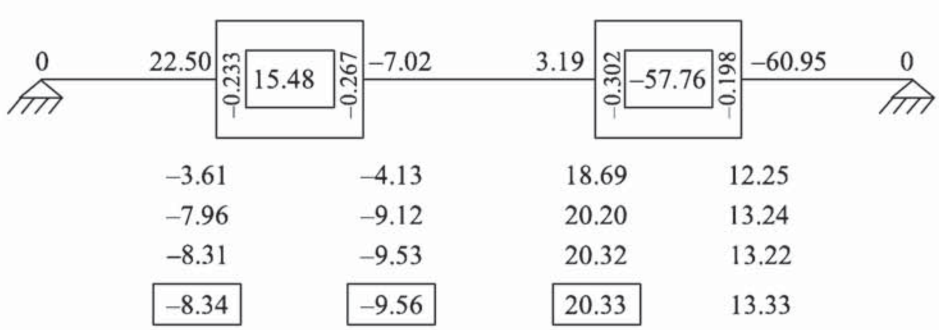
Rotation factor (RF) $= -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$

Table 3.16: Rotation factors

Joint	*Members*	*k*	Σk	*RF*
B	*BA*	$\frac{3EI}{3} = EI$	2.143 *EI*	–0.233
	BC	$\frac{4EI}{3.5} = 1.143\,EI$		–0.267
C	*CB*	1.143 *EI*	1.893 *EI*	–0.302
	CD	$\frac{3EI}{4} = 0.75\,EI$		–0.198

Rotation contributions are calculated and shown in Table 3.17 and final moments are calculated using Eqn. 3.2.

Table 3.17: Rotation contributions



Final Moment Calculations

Table 3.18: Final moment calculations

	A	*B*		*C*		*D*
FEM	0	22.5	−7.02	3.19	−60.95	0
Near end contribution	2 × 0	2(−8.34)	2(−9.56)	2 (20.33)	2(13.33)	2 × 0
Far end contribution	0	0	20.33	−9.56	0	0
Final moments	0	5.82	−5.82	34.29	−34.29	0

Example 3.6 Analyse the continuous beam shown in Figure 3.9 by Kani's method.

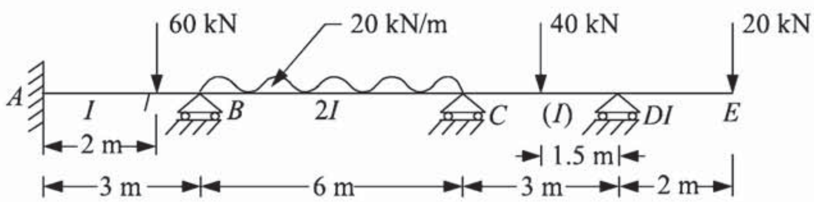


Figure 3.9: Continuous beam.

Solution ***Fixed End Moments***

$$M_{FAB} = -\frac{60 \times 2 \times 1^2}{3^2} = -13.33 \text{ kNm}$$

$$M_{FBA} = \frac{60 \times 2^2 \times 1}{3^2} = 26.67 \text{ kNm}$$

$$M_{FBC} = -\frac{20 \times 6^2}{12} = -60 \text{ kNm}$$

$$M_{FCB} = 60 \text{ kNm}$$

$$M_{FCD} = -\frac{40 \times 3.0}{8} = -15 \text{ kNm}$$

$$M_{FDC} = 15 \text{ kNm}$$

$$M_{FDE} = -2 \times 20 = -40 \text{ kNm}$$

Modification in FEM to account for rotation of *D*

$$M_{FDC} = -40 \text{ kNm (to balance joint } C)$$

$$M_{FCD} = -15 - 0.5(15 - 40) = -2.5 \text{ kNm}$$

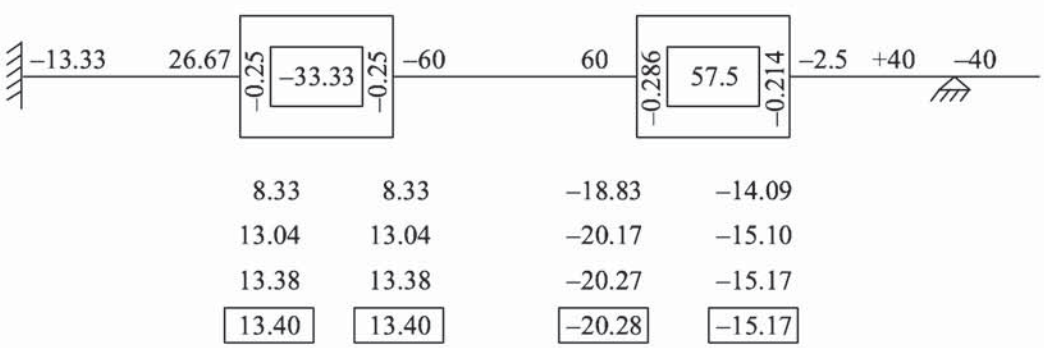
$$\text{Rotation factor (RF)} = -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.19: Rotation factors

Joint	*Members*	*k*	Σk	*RF*
B	*BA*	$\frac{4EI}{3}$	$\frac{8}{3}EI$	−0.25
	BC	$\frac{4E(2I)}{6} = \frac{4}{3}EI$		−0.25
C	*CB*	$\frac{4E(2I)}{6} = \frac{4}{3}EI$	2.33 *EI*	−0.286
	CD	$\frac{3EI}{3} = EI$		−0.214

Rotation contributions are calculated and shown in Table 3.20. The final moments are then calculated using Eqn. (3.2) and are shown in the table 3.21.

Table 3.20: Rotation contributions



Final Moment Calculations

Table 3.21: Final moment calculations

	A		*B*		*C*		*D*	*E*
FEM	−13.33	26.76	−60	60	−2.5	40	−40	–
Near end contribution	2 × 0	2(13.40)	2 × 13.40	2(−20.28)	2(−15.17)	0	–	–
Far end contribution	13.40	0	−20.28	13.40	0	–	–	–
Final	0.07	53.47	−53.47	32.84	−32.84	40	−40	0

3.3 ANALYSIS OF FRAMES WITHOUT LATERAL SWAY

If there is no sway in the frame, the analysis procedure is exactly the same as that for continuous beams, except that there may be more than two members meeting at the joint in the frames. While calculating rotation factors and rotation contributions, all the members meeting at the joints should be considered. This method is illustrated with the two examples (3.7 and 3.8) given below:

Example 3.7 Analyse the rigid frame shown in Figure 3.10 by Kani's method.

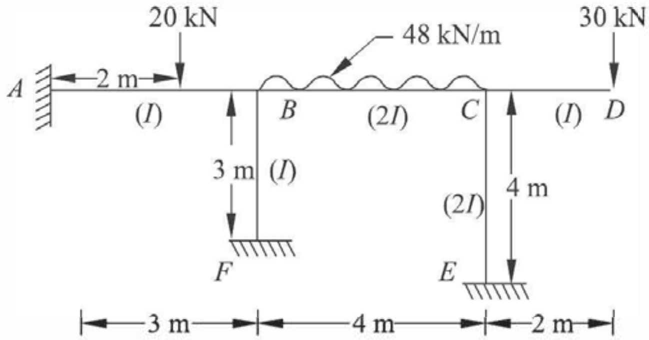


Figure 3.10: Rigid frame.

Solution ***Fixed End Moments***

$$M_{FAB} = -\frac{20 \times 2 \times 1^2}{3^2} = -4.44 \text{ kNm}$$

$$M_{FBA} = \frac{20 \times 2^2 \times 1}{3^2} = 8.89 \text{ kNm}$$

$$M_{FBC} = -\frac{48 \times 4^2}{12} = -64 \text{ kNm}$$

$$M_{FCB} = 64 \text{ kNm}$$

$$M_{FCD} = -30 \times 2 = -60 \text{ kNm}$$

$$M_{FBF} = M_{FFB} = M_{FCE} = M_{FEC} = 0$$

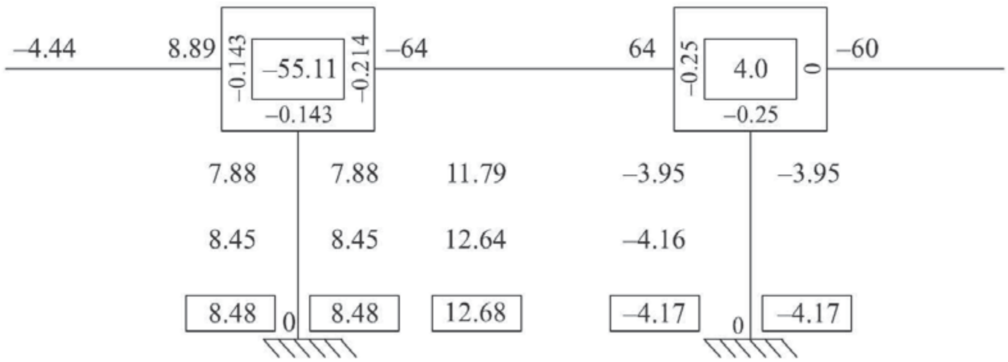
$$\text{Rotation factor (RF)} = -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.22: Rotation factors

Joint	*Members*	*k*	Σk	*RF*
B	*BA*	$\frac{4EI}{3}$	4.67 *EI*	–0.143
	BF	$\frac{4EI}{3}$		–0.143
	BC	$\frac{4E(2I)}{4} = 2EI$		–0.214
C	*CB*	2*EI*	4*EI*	–0.25
	CD	0		0
	CE	$\frac{4E(2I)}{4} = 2EI$		–0.25

Rotation contributions are calculated in tabular form and shown in Table 3.23.

Table 3.23: Rotation contributions



Final Moments

$$\text{Final moments} = \text{FEM} + 2\ (\text{Near end rotation contributions}) + (\text{Far end contributions})$$

$$M_{FAB} = -4.44 + 2 \times 0 + 8.48 = 4.04 \text{ kNm}$$

$$M_{BA} = 8.89 + 2 \times 8.48 + 0 = 25.85 \text{ kNm}$$

$$M_{BF} = 0 + 2 \times 8.48 + 0 = 16.96 \text{ kNm}$$

$$M_{BC} = -64 + 2 \times 12.68 - 4.17 = -42.81 \text{ kNm}$$

$$M_{CB} = 64 + 2(-4.17) + 12.68 = 68.34 \text{ kNm}$$

$$M_{CD} = -60 \text{ kNm}$$

$$M_{CE} = 0 + 2(-4.17) + 0 = -8.34 \text{ kNm}$$

$$M_{EC} = 0 + 2 \times 0 - 4.17 = -4.17 \text{ kNm}$$

$$M_{FB} = 0 + 2 \times 0 + 8.48 = 8.48 \text{ kNm}$$

Example 3.8 Analyse the symmetric frame shown in Figure 3.11(*a*) by Kani's method and indicate the final end moments on the sketch of the frame.

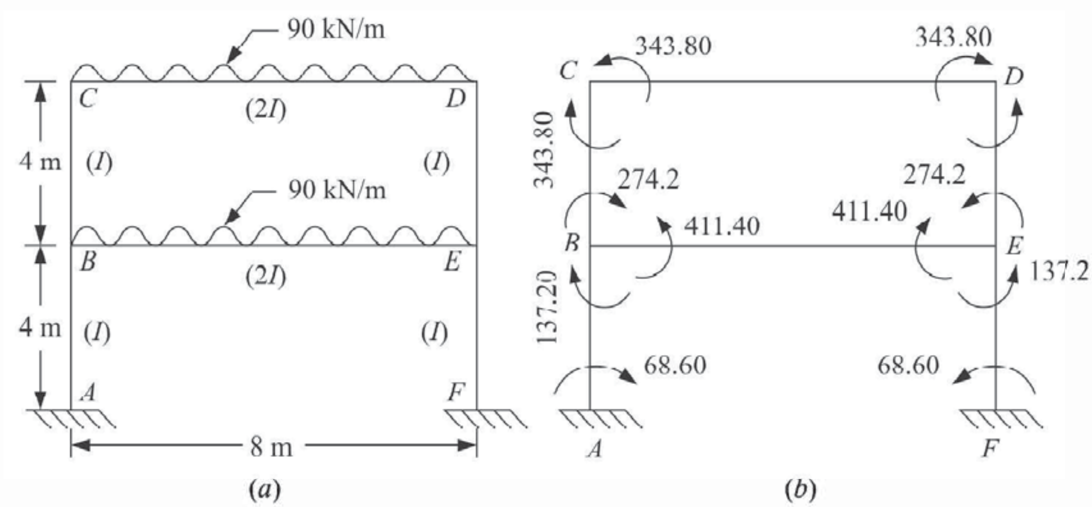


Figure 3.11(*a*): Symmetric frame. **(*b*)** End moments.

Solution

$$M_{FBA} = -\frac{90 \times 8^2}{12} = -480 \text{ kNm}$$
$$M_{FEB} = 480 \text{ kNm}$$
$$M_{FCD} = -480 \text{ kNm}$$
$$M_{FDC} = 480 \text{ kNm}$$

All other fixed end moments are zero.

$$\text{Rotation factor (RF)} = -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.24: Rotation factors

Joint	*Members*	*k*	Σk	*RF*
B	*BA*	$\frac{4EI}{4} = EI$	3*EI*	–0.167
	BE	$\frac{4E(2I)}{8} = EI$		–0.167
	BC	$\frac{4EI}{4} = EI$		–0.167
C	*CB*	$\frac{4EI}{4} = EI$	2*EI*	–0.25
	CD	$\frac{4E(2I)}{8} = EI$		–0.25
D	*DC*	$\frac{4E2I}{8} = EI$	2*EI*	–0.25
	DE	$\frac{4EI}{4} = EI$		–0.25
E	*ED*	$\frac{4EI}{4} = EI$	3*EI*	–0.167
	EB	$\frac{4E2I}{8} = EI$		–0.167
	EF	$\frac{4EI}{4} = EI$		–0.167

Rotation contributions are calculated in Table 3.25, starting from joint *C* and proceeding in the order *D*, *E*, *B* to complete the first cycle. After three more cycles, it has converged.

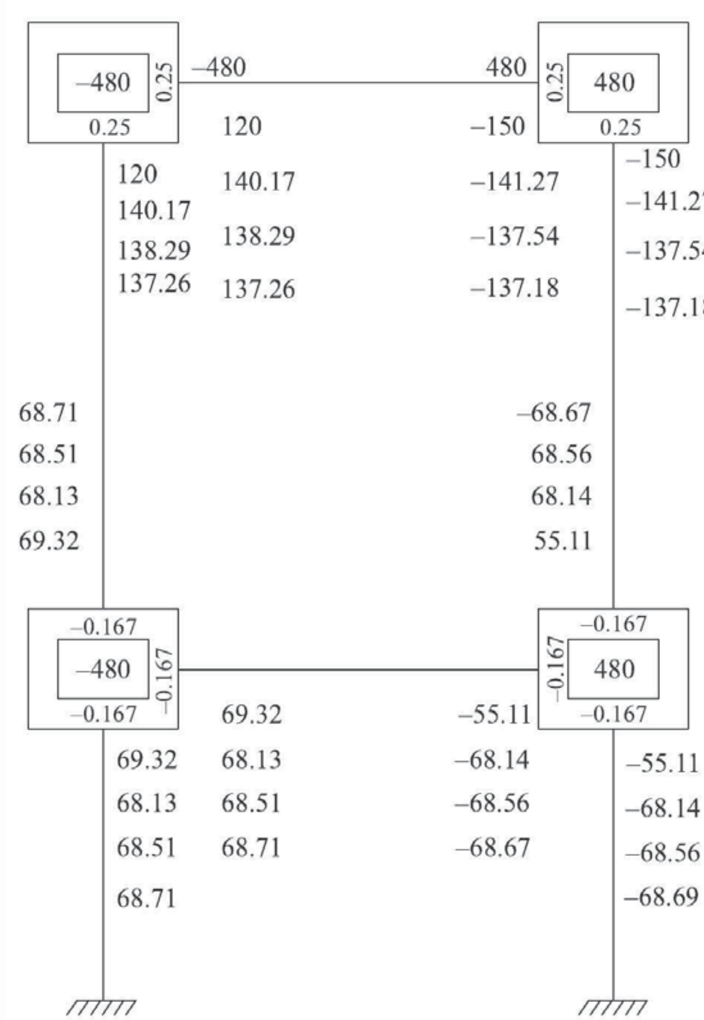
Final moments = FEM × 2 (Near end rotation contributions) + (Far end contributions)

$$M_{AB} = 0 + 2 \times 0 + 68.60 = 68.60 \text{ kNm}$$
$$M_{BA} = 0 + 2 \times 68.60 + 0 = 137.20 \text{ kNm}$$
$$M_{BE} = -480 + 2 \times 68.60 - 68.51 = -411.35 \text{ kNm}$$
$$M_{BC} = 0 + 2 \times 68.60 + 137.0 = 274.2 \text{ kNm}$$
$$M_{CB} = 0 + 2 \times 137.0 + 68.60 = 343.80 \text{ kNm}$$
$$M_{CD} = -480 + 2 \times 137 - 137.06 = 343.06 \text{ kNm}$$
$$M_{DC} = 480 + 2(-137.06) + 137 = 342.88 \text{ kNm}$$

$$M_{DE} = 0 - 2 \times 137.06 - 68.57 = 342.69 \text{ kNm}$$
$$M_{ED} = 0 - 2 \times 68.57 - 137.06 = 274.2 \text{ kNm}$$
$$M_{EB} = 480 - 2 \times 68.57 + 68.06 = 410.92 \text{ kNm}$$
$$M_{EF} = 0 - 2 \times 68.57 + 0 = -137.14 \text{ kNm}$$
$$M_{FE} = 0 + 2 \times 0 - 68.57 = -68.57 \text{ kNm}$$

But, for round-off errors, final moments are also symmetric. These values are rounded-off and indicated in Figure 3.11(*b*).

Table 3.25: Rotation contributions



3.4 ANALYSIS OF SYMMETRIC FRAMES TAKING ADVANTAGE OF SYMMETRY

Frames with symmetry do not have side sway. Apart from this, the bending moment values are going to be symmetric. Hence, it should be possible to make use of this and calculate the bending moment values only for one-half of the structure and using symmetry to write down for the other half. Two types of symmetric problems are encountered as shown in Figure 3.12.

(*a*) Line of symmetry passes though the columns. Such a case occurs when the number of bays are even.

(*b*) Line of symmetry passes through the mid-span of the beams. Such a case occurs when the number of bays are odd.

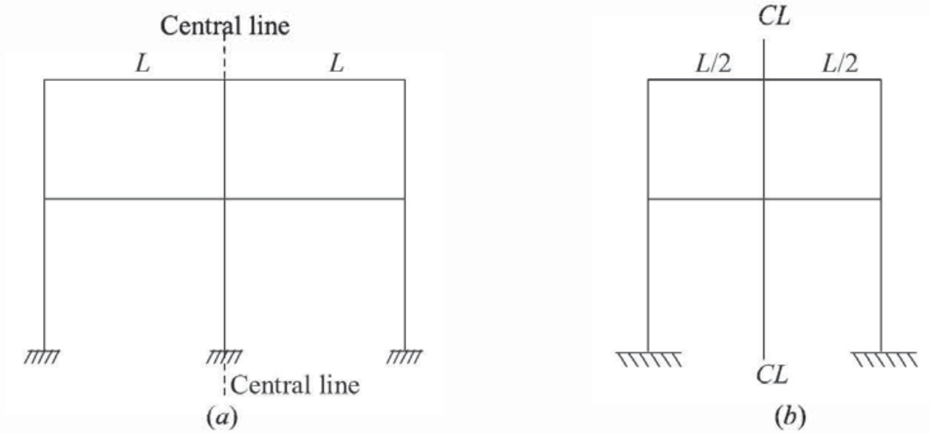


Figure 3.12(*a*): Symmetric line through columns. **(*b*)** Symmetric line through middle of beams.

3.4.1 Analysis of Symmetric Frames when Line of Symmetry Passes Through Columns

Because of symmetry, the joints on the line of symmetry do not rotate. Hence, they may be treated as fixed ends and only one half may be analysed. Using symmetry, the bending moment values are noted for the other half. Due to symmetry, there will not be any bending moment in the columns through which the line of symmetry passes.

Example 3.9 Analyse the symmetric frame shown in Figure 3.13(*a*). Make use of the symmetry for the analysis given that the moment of inertia of beams is twice that of the columns.

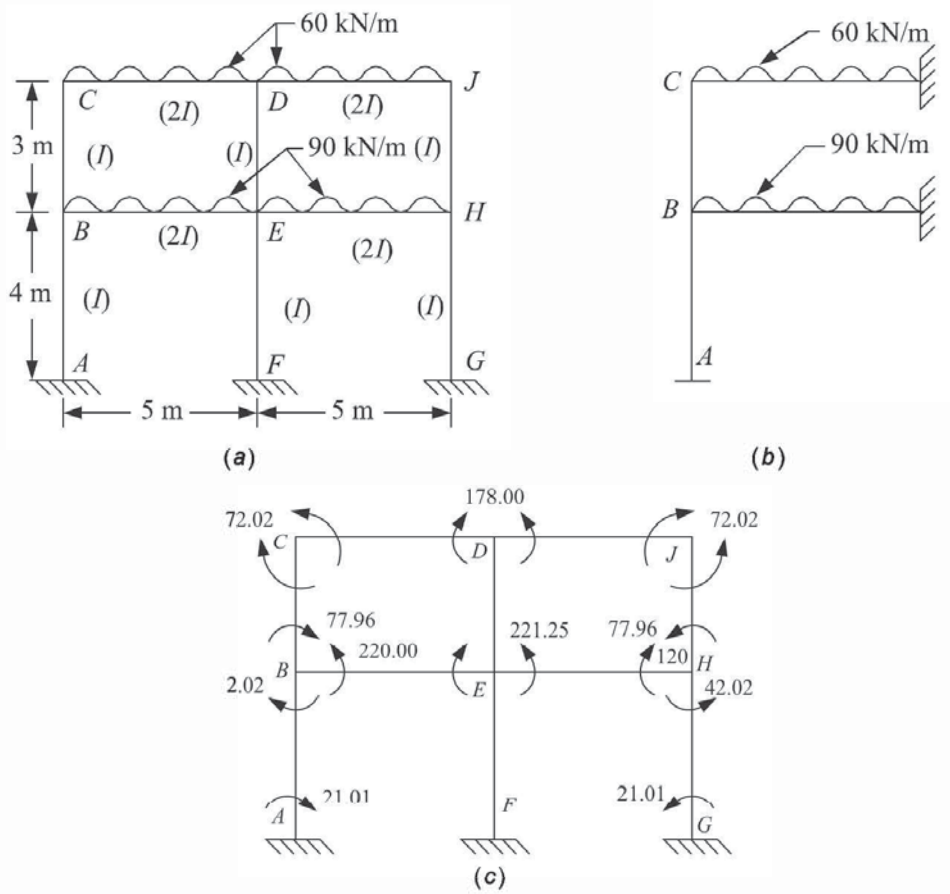


Figure 3.13(*a*): Symmetric frame. **(*b*)** Half the symmetric frame. **(*c*)** Final end moments.

Solution Since, the line of symmetry passes through columns *DEF*, joints *D* and *E* also will not rotate. Only half the frame as shown in Figure 3.13(*g*) may be considered for the analysis.

Fixed End Moments

$$M_{FCD} = -\frac{60 \times 5^2}{12} = -125\,\text{kNm}, \qquad M_{FDC} = 125\ \text{kNm}$$

$$M_{FBE} = -\frac{90 \times 5^2}{12} = -187.5\,\text{kNm}, \qquad M_{FEB} = 187.5\ \text{kNm}$$

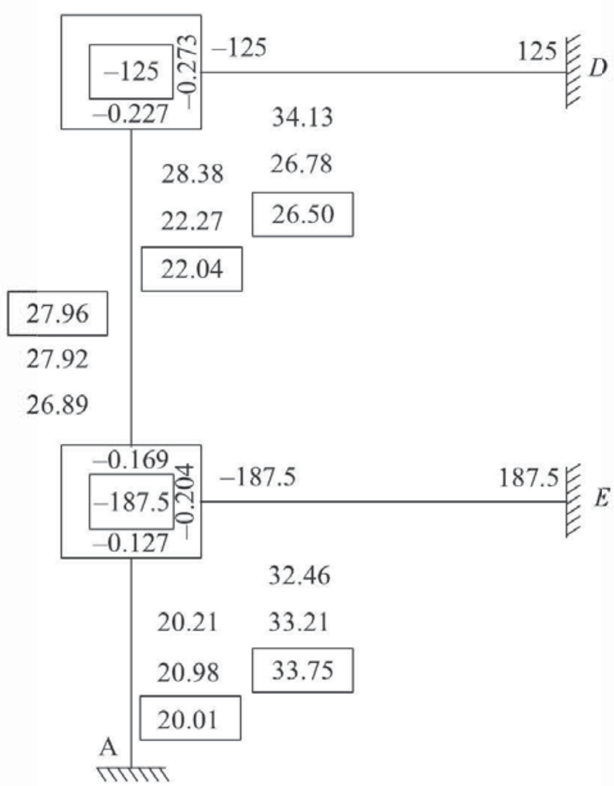
Fixed end moment at other ends = 0

Rotation factor (RF) = $-\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$

Table 3.26: Rotation factors

Joint	*Members*	*k*	Σ *k*	*RF*
B	*BA*	$\frac{4EI}{4} = EI$	3.93 *EI*	–0.127
	BE	$\frac{4E(2I)}{5} = 1.6EI$		–0.204
	BC	$\frac{4EI}{3} = 1.33\,EI$		–0.169
C	*CB*	$\frac{4EI}{3} = 1.33\,EI$	2.93 *EI*	–0.227
	CD	$\frac{4E(2I)}{5} = 1.6EI$		–0.273

Table 3.27: Rotation contributions



Analysis is carried out in Table 3.25 starting from joint *C* and then going to joint *B*:

Final Moments

$$M_{AB} = 0 + 2 \times 0 + 21.01 = 21.01 \text{ kNm}$$
$$M_{BA} = 0 + 2 \times 21.01 + 0 = 42.02 \text{ kNm}$$
$$M_{BE} = -187.5 + 2 \times 33.75 + 0 = -120 \text{ kNm}$$
$$M_{BC} = 0 + 2 \times 27.96 + 22.04 = 77.96 \text{ kNm}$$
$$M_{CB} = 0 + 2 \times 22.04 + 27.96 = 72.04 \text{ kNm}$$
$$M_{CD} = -125 + 2 \times 26.50 + 0 = -72.00 \text{ kNm}$$
$$M_{DC} = 125 + 2 \times 0 + 26.50 = 178 \text{ kNm}$$
$$M_{EB} = 187.5 + 2 \times 0 + 33.75 = 221.25 \text{ kNm}$$

Using symmetry, these values are marked on the other part of frame also (Refer Figure 3.13(*c*).

3.4.2 Analysis of Symmetric Frames when Line of Symmetry Passes through Mid-span of Beams

Because of symmetry, the rotation of mid-span of beams, through which the axis of symmetry passes, is zero. Considering only half the frame with fixed ends at the mid-span of beams, through which the line of symmetry passes, we can get the desired result for the entire frame.

Let *AB* shown in Figure 3.14(*a*) be a beam of symmetric frame through which the line of symmetry passes. Let the rotations be θ_A and θ_B at ends *A* and *B*, respectively.

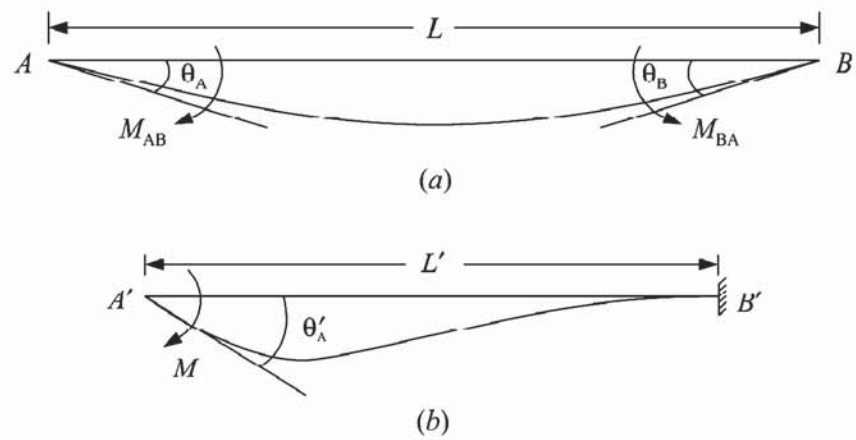


Figure 3.14(*a*): Beam with line of symmetry through mid-span.
(*b*) Beam with end *B′* fixed.

Due to symmetry,

$$\theta_A = \theta_B \quad \text{and} \quad M_{AB} = M_{BA} = M \text{ (numerically)}$$

Since, both the moments (M_{AB} and M_{BA}) cause clockwise rotations at *A*,

$$\theta_A = \frac{M_{AB}L}{3EI} + \frac{M_{BA}L}{6EI}$$

Since, $M_{AB} = M_{BA} = M$, we get,

$$\theta_A = M\left(\frac{L}{3EI}\right) + M\left(\frac{L}{6EI}\right)$$

$$= M\left(\frac{L}{2EI}\right) \qquad \ldots(i)$$

Now, consider a beam of span L' with end A' undergoing rotation while end B' is fixed (Refer Figure 3.14(b). Let moment M be aplied at joint A'. In such a case, we know,

$$\theta'_A = \frac{ML'}{4EI}, \text{ Since}, \frac{4EI}{L} \text{ is the stiffness of such a joint}$$

If the beam $A'B'$ has to replace beam AB,

$$\theta_A = \theta'_A$$

$$\frac{ML}{2EI} = \frac{ML'}{4EI} \quad \text{or} \quad \frac{2EI}{L} = \frac{4EI}{L'}$$

$$\frac{k}{2} = k'. \qquad \text{since stiffness } = \frac{4EI}{L}$$

Hence, if k is the stiffness of original beam, this beam may be replaced by another beam having its stiffness $\frac{k}{2}$, with fixed end at mid-span.

Example 3.10 Solve the Example 3.8 (Refer Figure 3.11) using symmetry.

Solution

$$M_{FBE} = -\frac{90 \times 8^2}{12} = -480 \text{ kNm}$$

$$M_{FEB} = 480 \text{ kNm}$$

$$M_{FCD} = -\frac{90 \times 8^2}{12} = -480 \text{ kNm}$$

$$M_{FDC} = 480 \text{ kNm}$$

Fixed end moments at all other ends are zero.

Now, using symmetry, only half the frame may be considered which has fixed ends at the mid-span of the beams (Refer Figure 3.15). Stiffness of CD' and BE' are modified as

$$\frac{1}{2} \times \frac{4E2I}{8} = 0.5\ EI$$

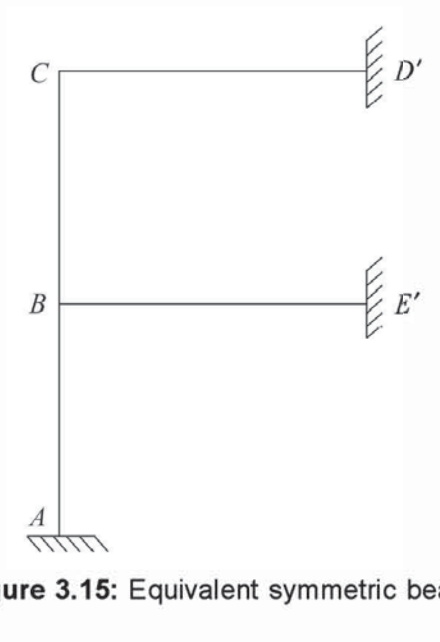


Figure 3.15: Equivalent symmetric beam.

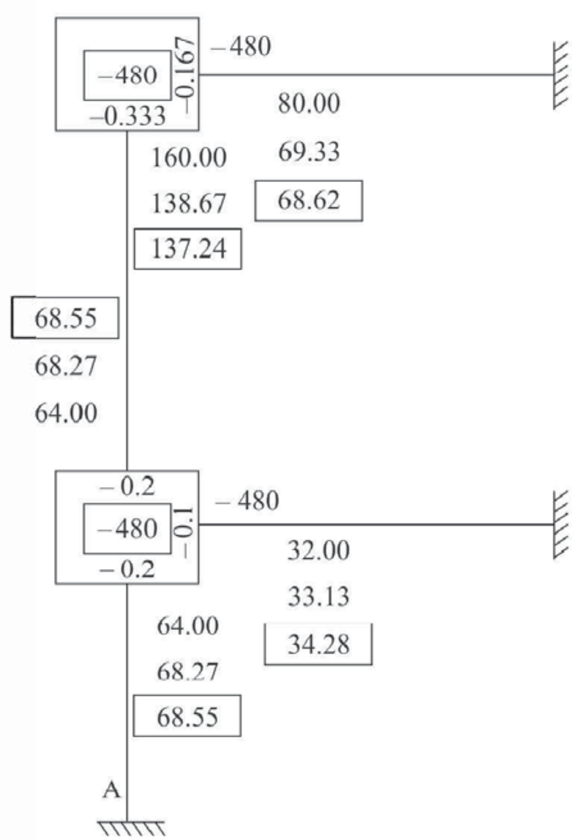
Rotation Factor

Table 3.28: Rotation factors

Joint	*Members*	*k*	Σ *k*	*RF*
C	CD′	0.5 *EI*	1.5 *EI*	– 0.167
	CB	$\frac{4\,EI}{4} = EI$		– 0.333
B	BA	$\frac{4\,EI}{4} = EI$	2.5 *EI*	– 0.2
	BE′	0.5 *EI*		– 0.1
	BC	$\frac{4\,EI}{4} = EI$		– 0.2

Rotation contributions are calculated and shown in Table 3.28. Joint *E* is considered first following which is joint *B*.

Table 3.29: Rotation contributions



Final Moments

$$M_{AB} = 0 + 2 \times 0 + 68.55 = 68.55 \text{ kNm} = -M_{FE}$$

$$M_{BA} = 0 + 2 \times 68.55 + 0 = 137.1 \text{ kNm} = -M_{EF}$$

$$M_{BE} = -480 + 2 \times 34.28 + 0 = 411.44 \text{ kNm} = -M_{EB}$$

$$M_{BC} = 0 + 2 \times 68.55 + 137.24 = 274.2 \text{ kNm} = -M_{ED}$$

$$M_{CB} = 0 + 2 \times 137.24 + 68.55 = 342.75 \text{ kNm} = -M_{DE}$$

$$M_{CD} = -480 + 2 \times 68.62 = -342.76 \text{ kNm} = -M_{DC}$$

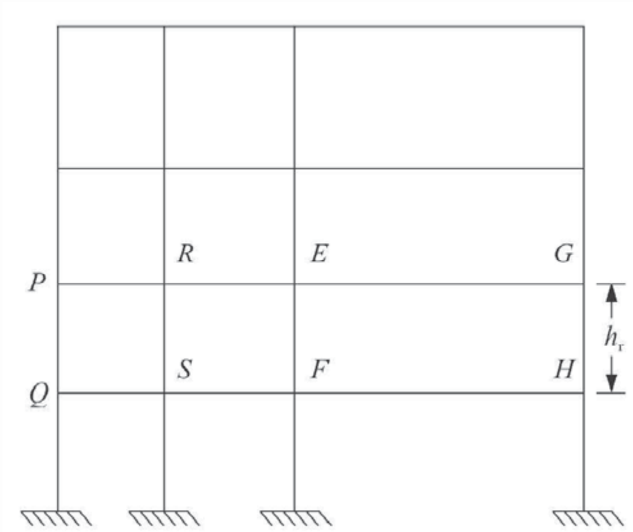
3.5 ANALYSIS OF FRAMES WITH SWAY

If a frame has unsymmetric geometry and/or unsymmetric load, it will sway laterally, causing relative displacements at the ends of columns. This lateral sway causes additional moments, which may be called *displacement contributions*. In the analysis of such frames which have sway, we come across two cases:

1. Heights of all columns in a storey are the same
2. Heights of columns in a storey are different.

3.5.1 Analysis of Frames with Sway when All Columns in a Storey have Same Height

Consider the frame shown in Figure 3.16(*a*). Usually, such frames are subjected to vertical loads. The wind loads which act horizontally are considered to act at joints. Hence, in columns, fixed end moment due to load is zero. Let *PQ*, *RS*, *EF* and *GH* be the columns in the *r*th storey and all of them of height h_r. Figure 3.16(*b*), shows the free body diagram of member *PQ*.



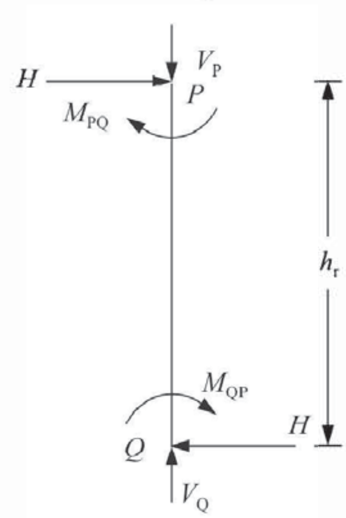


Figure 3.16(*a*): Typical frame having same column heights in all storey's.
(*b*) Freebody diagram of a column.

Let the horizontal force developed be H. Obviously, it is the shear force at any section in the member *PQ*, since all the horizontal forces act (if at all they do so) at joints only. Let M_{PQ} and M_{QP} be the final moments in *PQ* near joints *P* and *Q* respectively.

$$\Sigma M_P = 0, \text{ gives}$$

$$Hh_r + M_{PQ} + M_{QP} = 0$$

$$H = -\frac{\Sigma M_{PQ} + \Sigma M_{QP}}{h_r}$$

Summing up the horizontal forces (shear forces) in the *r*th storey, we get,

$$\Sigma H = -\frac{\Sigma M_{PQ} + \Sigma M_{QP}}{h_r}$$

i.e., $$S_r\, h_r = -(\Sigma M_{PQ} + \Sigma M_{QP}) \quad \ldots(3.4)$$

where S_r is the storey shear $= \Sigma H$

and the summation is over the columns in the storey. It may be noted that ΣM_{PQ} is the summation of end moments of all columns at the upper ends and ΣM_{QP} is the summation of the end moments at lower ends of the columns in the storey.

Let ΣM^*_{PQ} be the additional moment caused due to sway in member *PQ* at end *P*. Final moment M_{PQ} is given by

$$M_{PQ} = M_{FPQ} + 2M'_{PQ} + M'_{QP} + M^*_{PQ} \quad \ldots(3.5a)$$

where M_{FPQ} is the fixed end moment and M'_{PQ} and M'_{QP} are the near end and far end rotation contributions.

Similarly, $$M_{QP} = M_{FQP} + M'_{PQ} + 2M'_{QP} + M^*_{PQ} \quad \ldots(3.5b)$$

However, as we consider the case in which fixed end moments due to loads are zero in the columns,

$$M_{PQ} = 2M'_{PQ} + M'_{QP} + M^*_{PQ} \quad \ldots(3.6a)$$

and $$M_{QP} = M'_{PQ} + 2M'_{QP} + M^*_{QP} \quad \ldots(3.6b)$$

Adding Eqns. 3.6(*a*) and 3.6(*b*), we get,

$$M_{PQ} + M_{QP} = 3M'_{PQ} + 3M'_{QP} + M^*_{PQ} + M^*_{QP} \quad \ldots(3.7)$$

Due to relative displacements of ends, moments caused at the two ends are equal, $M^*_{PQ} = M^*_{QP}$

Hence, Eqn. 3.7 reduces to,

$$M_{PQ} + M_{QP} = 3(M'_{PQ} + M'_{QP}) + 2M^*_{PQ}$$

Summing up the above terms over all the columns in the storey, we get,

$$\Sigma M_{PQ} + \Sigma M_{QP} = 3(\Sigma M'_{PQ} + \Sigma M'_{QP}) + 2\Sigma M^*_{PQ}$$

or $$\Sigma M^*_{PQ} = \frac{1}{2}(\Sigma M_{PQ} + \Sigma M_{QP}) - \frac{3}{2}(\Sigma M'_{PQ} + \Sigma M'_{QP})$$

But, from Eqn. 3.4, $\Sigma M_{PQ} + \Sigma M_{QP} = -S_r h_r$

$$\Sigma M^*_{PQ} = -\frac{S_r h_r}{2} - \frac{3}{2}(\Sigma M'_{PQ} + \Sigma M'_{QP})$$

$$= -\frac{3}{2}\left[\frac{S_r h_r}{3} + (\Sigma M'_{PQ} + \Sigma M'_{QP})\right] \quad \ldots(3.8)$$

Since, $M^*_{PQ} = -\frac{6\,EI\Delta}{L^2} = -\frac{1.5k\,\Delta}{L}$ and Δ and L are same for all members,

$$\Sigma M^*_{PQ} = \left(-\frac{1.5\,\Delta}{L}\right)\Sigma k$$

$$\therefore \qquad \frac{M^*_{PQ}}{\Sigma M^*_{PQ}} = \frac{k}{\Sigma k}$$

$$M^*_{PQ} = \frac{k}{\Sigma k}\Sigma M^*_{PQ}$$

Substituting the values of ΣM^*_{PQ} from Eqn. 3.8, we get,

$$M^*_{PQ} = -\frac{3}{2}\left(\frac{k}{\Sigma k}\right)[S_r h_r + (\Sigma M'_{PQ} + \Sigma M'_{QP})] \qquad \dots(3.9)$$

The expression $-\frac{3}{2}\left(\frac{k}{\Sigma k}\right)$ is called the *displacement factor (DF)*. Thus,

$$\text{DF} = -\frac{3}{2}\left(\frac{k}{\Sigma k}\right)$$

where summation is over the number of columns in the storey. Thus, displacement contribution is given by

$$M^*_{PQ} = \text{DF}\left[\frac{S_r h_r}{3} + \left(\Sigma M'_{PQ} + \Sigma M'_{QP}\right)\right] \qquad \dots(3.10)$$

The term $\frac{S_r h_r}{3}$ is called the *storey moment.* Note that storey shear S_r can be easily assembled by considering the section through the entire storey and then writing the horizontal force equilibrium equation for the upper portion. Equation 3.10 may be stated as

Displacement contribution to a column

$$= \text{DF}\left(\begin{array}{l}\text{Storey moment} + \sum \text{Rotation contribution at top and bottom} \\ \qquad\qquad\qquad\quad \text{ends of the columns in the storey}\end{array}\right) \qquad \dots(3.11)$$

When displacement contributions are to be considered, Eqn. 3.3 for calculating rotation contribution gets modified to

$$M'_{AB} = RF\left(\Sigma M_{FAB} + \Sigma M'_{BA} + \Sigma M^*_{PQ}\right) \qquad \dots(3.12)$$

Eqns. 3.11 and 3.12 help in evolving a distribution procedure that will take care of sway.

1. Find fixed end moments in all the members.
2. Find rotation factors at all the joints which are going to rotate.
3. Find displacement factors for the columns in each storey.
4. Prepare Kani's distribution table.
5. To start with, unknown values of all rotation contributions and displacement contributions are taken equal to zero.
6. As and when the rotation contributions and displacement contributions are available, those values are considered.
7. Kani's procedure is applied joint by joint to calculate rotation contributions, till all joints are handled.
8. After all joints are handled and before going for the next cycle, displacement contributions are calculated using Eqn. 3.11.

9. Start with next cycle. At this stage, displacement contributions are also available. Hence, it should be considered while calculating the rotation contributions.

   Rotation contributions = $\text{RF}\left(\Sigma M_{\text{FAB}} + \Sigma M'_{\text{BA}} + \Sigma M^{*}_{\text{PQ}}\right)$

where $\Sigma M'_{\text{FAB}}$ is sum of fixed end moments at the joint,

$\Sigma M'_{\text{BA}}$ is the far end contributions of all the members meeting at the joint.

and ΣM^{*}_{PQ} is the displacement contributions of all the columns meeting at the joint.

10. Repeat the cycle till the rotation and displacement contributions are negligible.
11. Assemble the final moments using Eqn. 3.5.

Final moment = Fixed end moment + 2 (Near end rotation contribution)
+ Far end rotation contribution + Displacement contribution

However, note that there are no displacement contribution for beams and no fixed end moments for columns.

Example 3.11 Using Kani's rotational contribution method, analyse the frame shown in Figure 3.17. Moment of inertia of the members are shown encircled near the members.

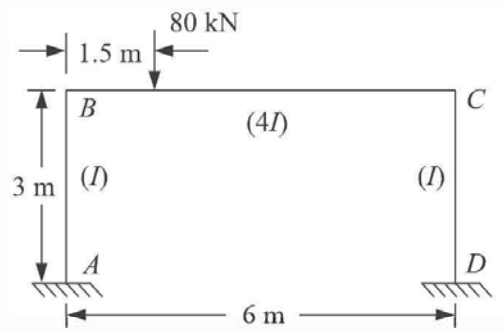


Figure 3.17: Given frame.

Solution ***Fixed End Moments***

$$M_{\text{FBC}} = -\frac{80 \times 1.5 \times 4.5^2}{6^2} = -67.5\ \text{kNm}$$

$$M_{\text{FCB}} = \frac{80 \times 1.5^2 \times 4.5}{6^2} = 22.5\ \text{kNm}$$

Fixed end moments in both the columns are zero.

$$\text{Rotation factor (RF)} = \frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.30: Rotation factors

Joint	*Members*	*k*	Σ *k*	*RF*
B	*BA*	$\frac{4EI}{3}$	4 *EI*	−0.167
	BC	$\frac{4E(4I)}{6} = \frac{8}{3}EI$		−0.333
C	*CB*	$\frac{4E(4I)}{6} = \frac{8}{3}EI$	4 *EI*	−0.333
	CD	$\frac{4EI}{3}$		−0.167

$$\text{Displacement factor (DF)} = -\frac{3}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.31: Displacement factors

Storey	*Members*	*k*	Σ *k*	*DF*
1	*AB*	$\frac{4EI}{3}$	$\frac{8EI}{3}$	–0.75
	DC	$\frac{4EI}{3}$		– 0.75

Distribution is carried out in Table 3.32. The process starts from joint *B* to joint *C*. After finding the rotation contributions at *B* and *C*, the displacement contributions are to be found. In this problem, horizontal equilibrium of the first storey, after taking a section through it, shows storey shear.

$$S_r = \Sigma H = 0$$

Therefore, storey moment $= \frac{1}{3} S_r h_r = 0$

Displacement contribution = DF (Σ Rotation contributions at top and bottom of columns *AB* and *CD*).

Rotation contribution after first cycle = –0.75[11.25 + 0 –7.5 + 0] = –2.81

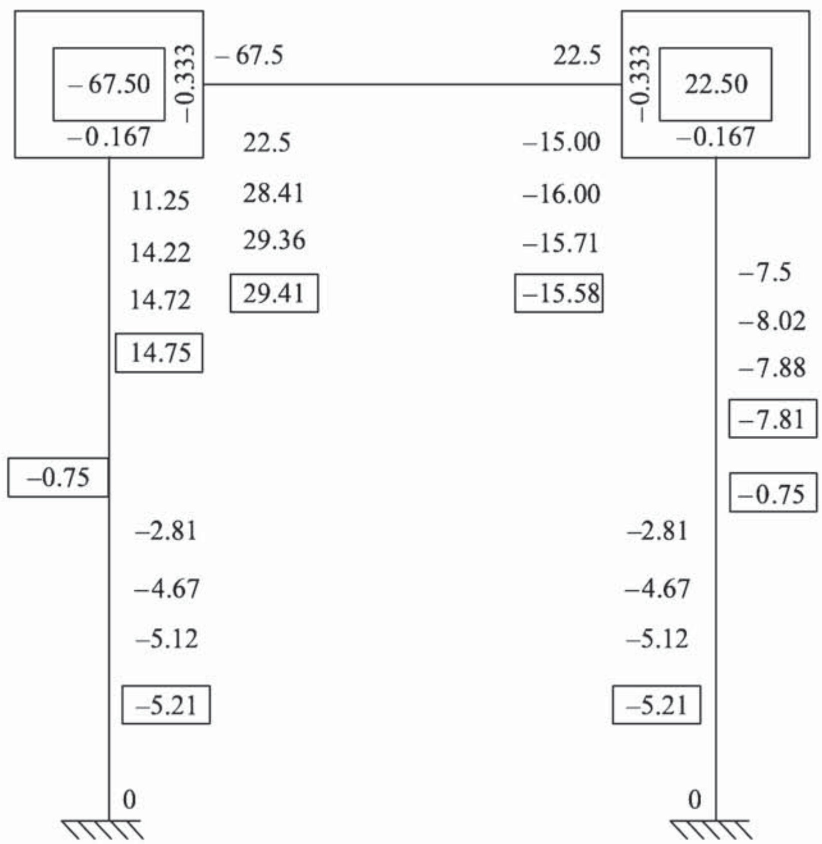
This is written in the middle of the column.

Then, the second cycle is started.

$$M'_{BA} = -0.167[-67.5 + 0 - 15.00 - 2.81] = 14.22$$

and $M'_{BC} = -0.333[-67.5 + 0 - 15.00 - 2.81]$, since there is no displacement in the beam $= 28.47$

Table 3.32: Rotation and displacement contributions



$M'_{BC} = 28.47$

At joint C, we again calculate the displacement contributions. At the end of the fourth cycle, changes in rotation and displacement contributions are negligible. Hence, distribution procedure is stopped.

Final Moments

Using Eqn. 3.5, these are assembled.

$$M_{AB} = 0 + 2 \times 0 + 14.75 - 5.29 = 9.46 \text{ kNm}$$
$$M_{BA} = 0 + 2 \times 14.75 + 0 - 5.21 = 24.29 \text{ kNm}$$
$$M_{BC} = -67.5 + 2 \times 29.41 - 15.58 + 0 = -24.26 \text{ kNm}$$
$$M_{CB} = 22.5 - 2 \times 15.58 + 29.41 + 0 = 20.75 \text{ kNm}$$
$$M_{CD} = 0 + 2(-7.81) + 0 + (-5.21) = -20.83 \text{ kNm}$$
$$M_{DC} = 0 + 2 \times 0 - 7.81 - 5.21 = -13.02 \text{ kNm}$$

Example 3.12 Analyse the rigid jointed frame shown in Figure 3.18 by Kani's method.

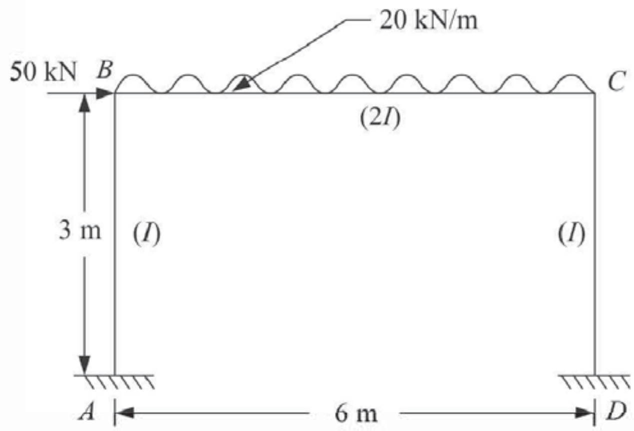


Figure 3.18: Rigid jointed frame.

Solution ***Fixed End Moments***

$$M_{FBC} = -\frac{20 \times 6^2}{12} = -60 \text{ kNm}$$

$$M_{FCB} = 60 \text{ kNm}$$

Other FEMs are zero.

$$\text{Rotation factor (RF)} = -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.33: Rotation factors

Joint	*Members*	*k*	Σk	*RF*
B	BA	$\frac{4EI}{3}$	$\frac{8}{3}EI$	−0.25
	BC	$\frac{4E(2I)}{6} = \frac{4}{3}EI$		−0.25
C	CB	$\frac{4E(2I)}{6} = \frac{4}{3}EI$	$\frac{8}{3}EI$	−0.25
	CD	$\frac{4EI}{3}$		−0.25

Displacement factors (DF) = $-\dfrac{3}{2}\left(\dfrac{k}{\Sigma k}\right)$

Table 3.34: Distribution factors

Storey	*Members*	*k*	Σ *k*	*DF*
1	*AB*	$\frac{4EI}{3}$	$\frac{8EI}{3}$	−0.75
	DC	$\frac{4EI}{3}$		−0.75

Taking the section through columns of first storey and considering the horizontal equilibrium of the upper portion, we get,

Storey shear = 50 kN

Therefore, storey moment = $\dfrac{S_r h_r}{3} = \dfrac{50 \times 3}{3} = 50 \text{ kNm}$

Distribution Procedure

It first starts from end *B* and then proceeds to end *C*. After obtaining the rotation contributions at *C*, displacement contributions are to be found (Ref. Table 3.35).

From Eqn. 3.11,

Displacement contribution = DF $\left(\begin{array}{r}\text{Storey moment} + \sum \text{Rotation contributions at top} \\ \text{and bottom end of the columns}\end{array}\right)$

Therefore, for column *AB*, displacement contribution

$= -0.75\ (50 + 15 + 0 - 18.75 + 0)$

$= -34.69$

For column *CD*, displacement factor $DF = -0.75$

Therefore, distribution contribution = −34.69

This completes the first cycle. In the second cycle, rotation contribution at joint *B*

= RF(−60 + (0 − 18.75) −34.69)

= RF(−113.44)

= 28.36, for both beam and column, since RF = −0.25 for both.

Similarly, rotation contributions at joint *C*

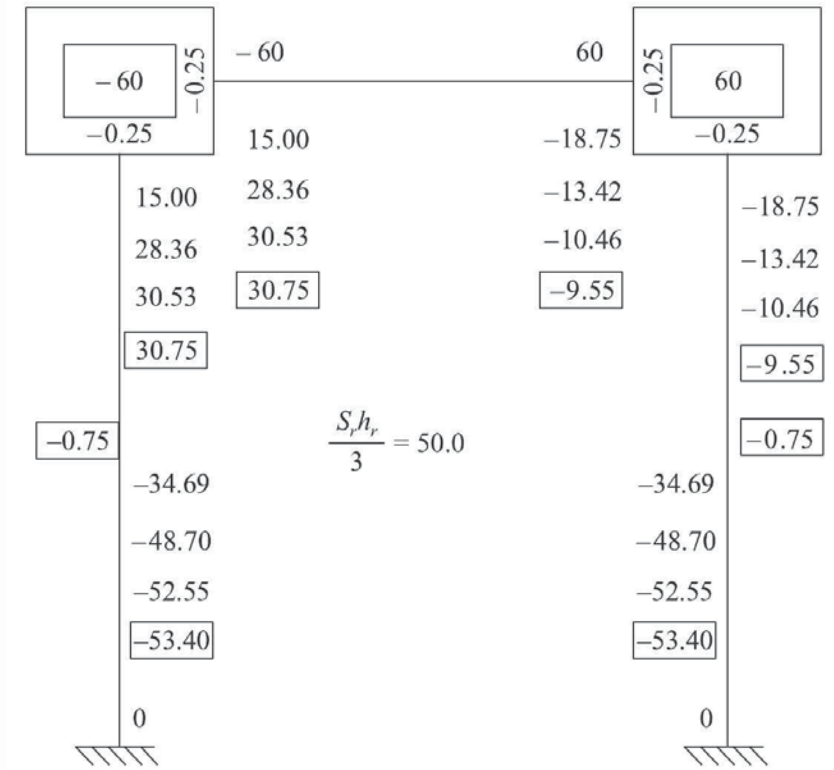
= RF(60 + (28.36 + 0) − 34.69)

= RF(53.67)

= −13.42, Since RF = −0.25 for both the members.

Schematic calculation is shown in Table 3.35

Table 3.35: Rotation and displacement contributions



Final Moments

Equation 3.5 is used to find the final end moments.

$$M_{AB} = 0 + 2 \times 0 + 30.75 - 53.40 = -22.65 \text{ kNm}$$
$$M_{BA} = 0 + 2 \times 30.75 + 0 - 53.40 = 8.1 \text{ kNm}$$
$$M_{BC} = -60 + 2 \times 30.75 - 9.55 + 0 = -8.05 \text{ kNm}$$
$$M_{CB} = 60 - 2 \times 9.55 + 30.75 + 0 = 71.65 \text{ kNm}$$
$$M_{CD} = 0 + 2\,(-9.55) + 0 - 53.40 = -72.50 \text{ kNm}$$
$$M_{DC} = 0 + 2 \times 0 - 9.55 - 53.40 = -62.95 \text{ kNm}$$

Example 3.13 Analyse the frame shown in Figure 3.19 by Kani's method.

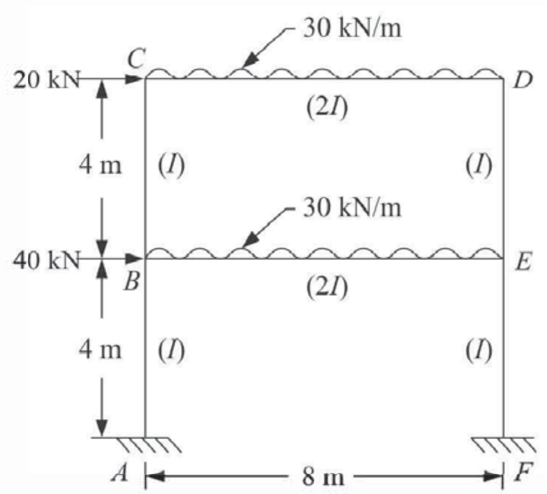


Figure 3.19: Given frame.

Solution ***Fixed End Moments***

Fixed end moments in columns = 0

$$M_{FBE} = -\frac{30 \times 8^2}{12} = -160\,\text{kNm}$$

$$M_{FEB} = 160\text{ kNm}$$

$$M_{FCD} = -160\text{ kNm}$$

$$M_{FDC} = 160\text{ kNm}$$

$$\text{Rotation factors (RF)} = -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.36: Rotation factors

Joint	*Members*	*k*	Σk	*RF*
B	BA	$\frac{4EI}{4} = EI$	3 EI	–1/6
	BE	$\frac{4E(2I)}{8} = EI$		–1/6
	BC	$\frac{4EI}{4} = EI$		–1/6
C	CB	$\frac{4EI}{4} = EI$	2 EI	–0.25
	CD	$\frac{4E(2I)}{8} = EI$		–0.25
D	DC	$\frac{4E(2I)}{8} = EI$	2 EI	–0.25
	DE	$\frac{4EI}{4} = EI$		–0.25
E	ED	EI	3 EI	–1/6
	EB	EI		–1/6
	EF	EI		–1/6

$$\text{Displacement factors (DF)} = -\frac{3}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.37: Displacement factors

Storey	*Members*	*k*	Σk	*DF*
I	AB	EI	2 EI	–0.75
	FE	EI		–0.75
II	BC	EI	2 EI	–0.75
	ED	EI		–0.75

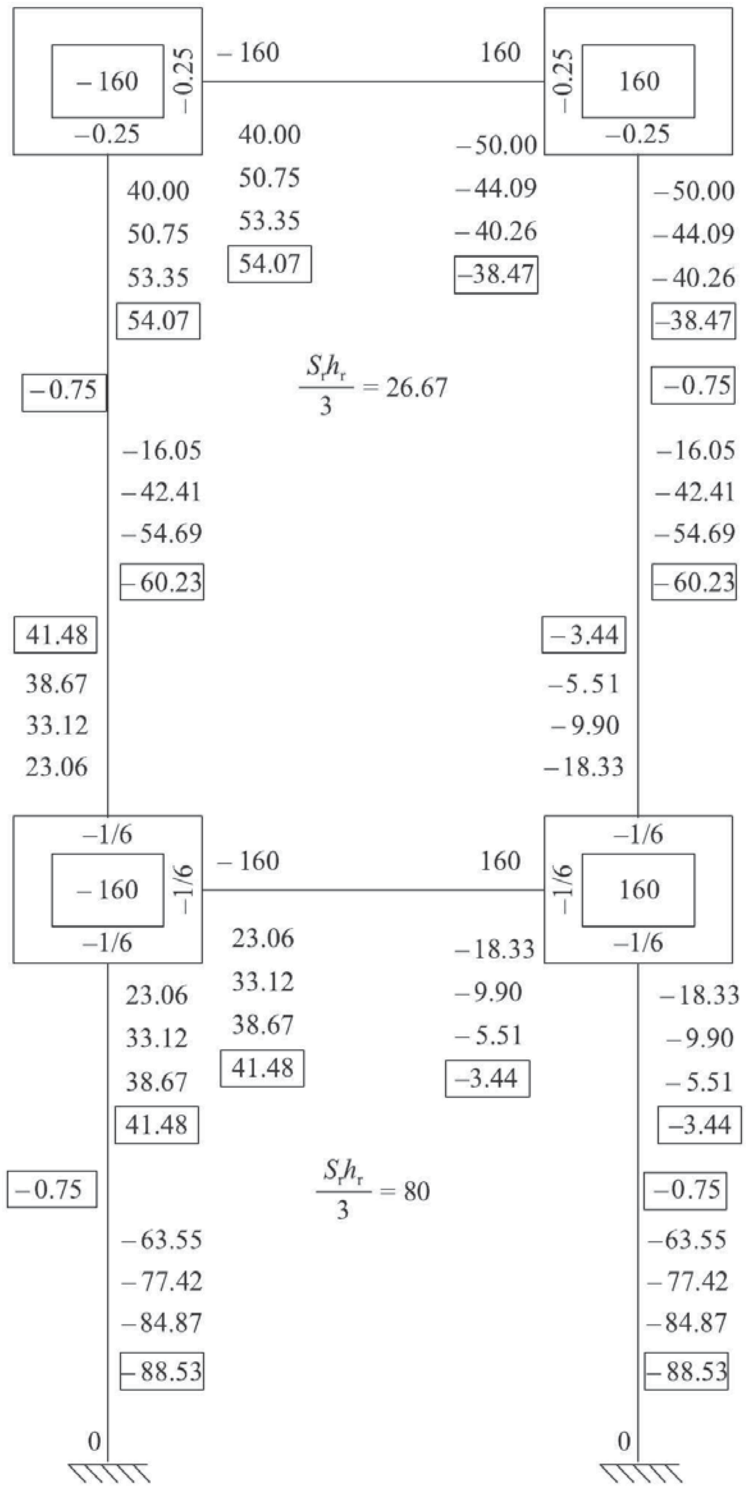
Storey Moments

Taking the section through first storey and considering the equilibrium of horizontal forces, we find that storey shear, $Sr_1 = 20 + 40 = 60$ kN

Considering the section through second storey, we find that $Sr_2 = 20$ kN

$\therefore$ Storey moments in I storey $= \dfrac{60 \times 4}{3} = 80$ kNm and in II storey $= \dfrac{20 \times 4}{3} = 26.67$ kNm

Table 3.38: Rotation and displacement contributions



Distribution procedure is started from joint C form which joint D, E, B are considered in the order. After finding the rotation contributions at all the four joints, displacement contributions are found. The schematic calculations are shown in Table 3.38. Iterative porocedure is stopped after four cycles and then the final moments are calculated.

Final Moments

$$M_{AB} = 0 + 2 \times 0 + 41.48 - 88.53 = -47.05 \text{ kNm}$$

$$M_{BA} = 0 + 2 \times 41.48 + 0 - 88.53 = -5.57 \text{ kNm}$$

$$M_{BE} = -160 + 2 \times 41.48 - 3.44 + 0 = -80.48 \text{ kNm}$$

$$M_{BC} = 0 + 2 \times 41.48 + 54.07 - 60.23 = 89.39 \text{ kNm}$$

$$M_{CD} = -160 + 2 \times 54.07 - 38.47 + 0 = -90.33 \text{ kNm}$$

$$M_{DC} = 160 + 2 \times 38.47 + 54.07 + 0 = 137.13 \text{ kNm}$$

$$M_{DE} = 0 + 2\,(-38.47) - 3.44 - 60.23 = -140.61 \text{ kNm}$$

$$M_{ED} = 0 + 2\,(-3.44) - 38.47 - 60.23 = -105.58 \text{ kNm}$$

$$M_{EF} = 0 + 2\,(-3.44) - 88.53 = -95.41 \text{ kNm}$$

$$M_{EB} = 160 - 2 \times 3.44 + 41.48 + 0 = 194.6 \text{ kNm}$$

$$M_{FE} = 0 + 2 \times 0 - 3.44 - 88.53 = -91.97 \text{ kNm}$$

3.5.2 Analysis of Frames with Sway when Columns in a Storey have Different Heights

If the columns are of different heights, choose a convenient height as the storey height. Then, Eqn. 3.8 may be modified as

$$\Sigma M^*_{PQ}\left(\frac{h_r}{h_{PQ}}\right) = -\frac{3}{2}\left[\frac{S_r h_r}{3} + \Sigma\left(M'_{PQ} + M'_{QP}\right)\right]$$

Denoting $\left(\dfrac{h_r}{h_{PQ}}\right)$ as C_{PQ}, we get,

$$\Sigma C_{PQ}\, M^*_{PQ} = -\frac{3}{2}\left[\frac{S_r h_r}{3} + \Sigma C_{PQ}\left(M'_{PQ} + M'_{QP}\right)\right]$$

But
$$M^*_{PQ} = -\left(\frac{6\,EI\,\Delta}{h^2_{PQ}}\right) = -\frac{1.5\,k\Delta}{h_{PQ}}$$

i.e.,
$$M^*_{PQ} \propto \frac{k}{h_{PQ}}$$

or $$\propto k\left(\frac{h_r}{h_{PQ}}\right), \text{ Since, } h_r \text{ is constant}$$

or $$\propto k\ C_{PQ}$$

$$\therefore \quad \frac{M^*_{PQ}}{\sum M^*_{PQ}} = \frac{kC_{PQ}}{\left[\sum C_{PQ}\ kC_{PQ}\right]} = \frac{kC_{PQ}}{\sum C^2_{PQ}\ k}$$

$$\therefore \quad M^*_{PQ} = -\frac{3}{2}\left[\frac{kC_{PQ}}{\sum C^2_{PQ}\ k}\left[\frac{S_r h_r}{3} + \sum\left(M'_{PQ} + M'_{QP}\right)C_{PQ}\right]\right]$$

$$\therefore \quad \text{Displacement factor (DF)} = -\frac{3}{2}\left(\frac{kC_{PQ}}{\Sigma C^2_{PQ}\ k}\right) \qquad \ldots(3.13)$$

$$\text{Displacement contribution } M^*_{PQ} = \text{DF}\left[\frac{S_r h_r}{3} + \sum\left(M'_{PQ} + M'_{QP}\right)C_{PQ}\right] \qquad \ldots(3.14)$$

With the above modifications for displacement factor and displacement contributions, frames with different heights can be analysed.

Example 3.14 Analyse the portal frame shown in Figure 3.20, by Kani's method.

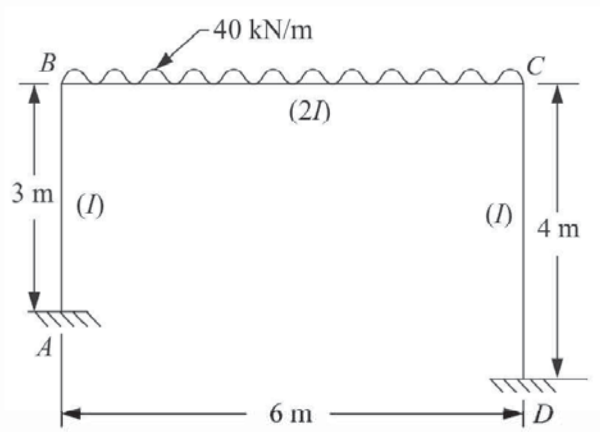


Figure 3.20: Portal frame.

Solution ***Fixed End Moments***

Fixed end moments in columns = 0

i.e., $$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

$$M_{FBC} = -\frac{40 \times 6^2}{12} = -120 \text{ kNm}$$

$$M_{FCB} = \frac{40 \times 6^2}{12} = 120 \text{ kNm}$$

$$\text{Rotation factor (RF)} = -\frac{1}{2}\left(\frac{k}{\Sigma k}\right)$$

Table 3.39: Rotation factors

Joint	*Members*	*k*	Σk	*RF*
B	*BA*	$\frac{4EI}{3}$	$\frac{8EI}{3}$	– 0.25
	BC	$\frac{4E(2I)}{6} = \frac{4EI}{3}$		– 0.25
C	*CB*	$\frac{4E(2I)}{6} = \frac{4EI}{3}$	$\frac{7EI}{3}$	– 0.286
	CD	$\frac{4EI}{4} = EI$		– 0.214

$$\text{Displacement factors (DF)} = -\frac{3}{2}\left(\frac{kC_{PQ}}{\sum C_{PQ}^2 k}\right)$$

Let the storey height h_r be 3m

$$C_{AB} = 1.0$$

$$C_{CD} = \frac{3}{4} = 0.75$$

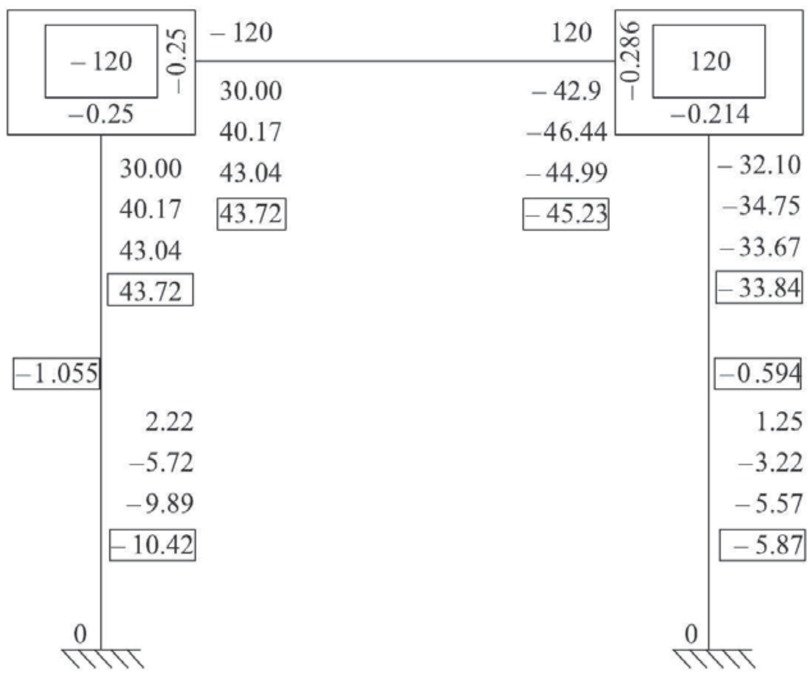
Table 3.40: Displacement factors

Storey	*Members*	*k*	C_{PQ}	$\sum C_{PQ}^2 k$	*DF*
I	*AB*	$\frac{4EI}{3} = 1.3333\,EI$	*1.0*	1.8955	– 1.055
	DC	$\frac{4EI}{4} = EI$	0.75		– 0.594

Distribution

Storey moment is zero. From joint *B*, the distribution procedure is started. The schematic calculations are shown in Table 3.41. Displacement contributions are calculated using Eqn. 3.24.

Table 3.41: Rotations and displacement contributions



Final Moments

$$M_{AB} = 0 + 2 \times 0 + 43.72 - 10.42 = 33.30$$
$$M_{BA} = 0 + 2 \times 43.72 - 10.42 = 77.02$$
$$M_{BC} = -120 + 2 \times 43.72 - 45.23 = -77.79$$
$$M_{CB} = 120 - 2 \times 45.23 + 43.72 + 0 = 73.26$$
$$M_{CD} = 0 + 2\,(-33.84) - 5.87 = -73.55$$
$$M_{DC} = 0 + 2 \times 0 - 33.84 - 5.87 = -39.71$$

SUMMARY

1. Kani's method of rotation contribution is explained taking clockwise end moments and rotations as positive values.
2. The rotation factor is $-\frac{1}{2}\frac{K_{AB}}{\Sigma K_{AB}}$, where summation is over the members meeting at the joint under consideration.
3. After finding rotation contributions the final moments at an end of member may be found by the formula.

   Final moment = FEM + 2 × Rotation contribution at near end + Rotation contribution at far end.
4. In symmetric frames, if the line of symmetry passes through columns, then the central columns are not subjected to any moment. In such cases, half the frame may be considered for analysis treating beams as fixed at central columns.

5. Symmetric frames with the line of symmetry through mid-span of beams may be analysed taking half the frame with fixed ends for beams on line of symmetry and stiffness of them halved.
6. If there is sway in the frame, displacement contributions should also be considered. The displacement contribution is given by:

$$M^*_{AB} = DF\left[\frac{S_r h_r}{3} + \left(\Sigma M'_{PQ} + \Sigma M'_{QP}\right)\right]$$

where $\frac{S_r h_r}{3}$ is storey moment and DF is the displacement contribution = $-\frac{3}{2}\frac{k}{\Sigma k}$, the summation being over the columns in the storey.
7. If columns in a storey have different heights, the displacement contribution is given by:

$M^*_{PQ} = DF\left[\frac{S_r h_r}{3} + \Sigma\left(M'_{PQ} + M'_{QP}\right)C_{PQ}\right]$, where h_r is the selected storey height and $C_{PQ} = \frac{h_r}{h_{PQ}}$.

MULTIPLE CHOICE QUESTIONS

Select the correct option:

1. The rotation factor for a member *AB* at a joint is:

(a) $-\frac{1}{2}\frac{k}{\Sigma k}$ (b) $\frac{1}{2}\frac{k}{\Sigma k}$

(c) $-\frac{3}{2}\frac{k}{\Sigma k}$ (d) $\frac{3}{2}\frac{k}{\Sigma k}$ **[Ans:** 1. (a)]

2. The displacement factor as used in Kani's method is:

(a) $-\frac{1}{2}\frac{k}{\Sigma k}$ (b) $\frac{1}{2}\frac{k}{\Sigma k}$

(c) $-\frac{3}{2}\frac{k}{\Sigma k}$ (d) $\frac{3}{2}\frac{k}{\Sigma k}$ **[Ans:** 2. (c)]

3. Storey moment in terms of storey shear S_r and storey height h_r is:

(a) $S_r h_r$ (b) $-S_r h_r$

(c) $\frac{1}{3}S_r h_r$ (d) $-\frac{1}{3}S_r h_r$ **[Ans:** 3. (c)]

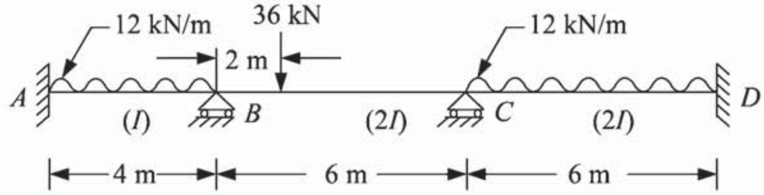
4. In a frame, if the line of symmetry passes through the midspan of beams, the symmetric beam of stiffness *k* may be replaced by a beam with fixed end at mid-span and having stiffness:

(a) *k* (b) –*k*

(c) *k*/2 (d) *K* **[Ans:** 4. (c)]

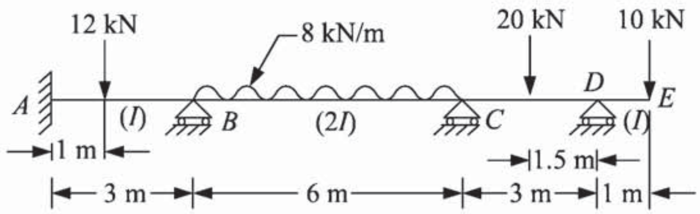
EXERCISES

1. Analyse the continuous beam shown in figure given below by Kani's method.



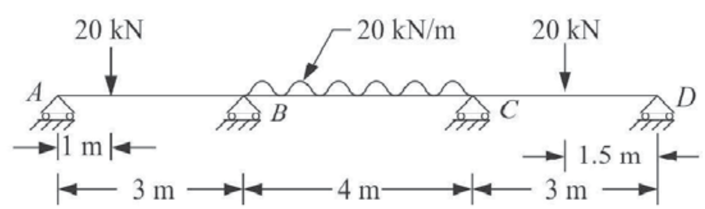
[Ans: M_{AB} = –13.46 kNm; M_{BA} = –M_{BC} = 21.08 kNm; M_{CB} = –M_{CD} = 27.70 kNm; M_{DC} = 40.15 kNm.]

2. Analyse the continuous beam shown in figure given below by Kani's method.



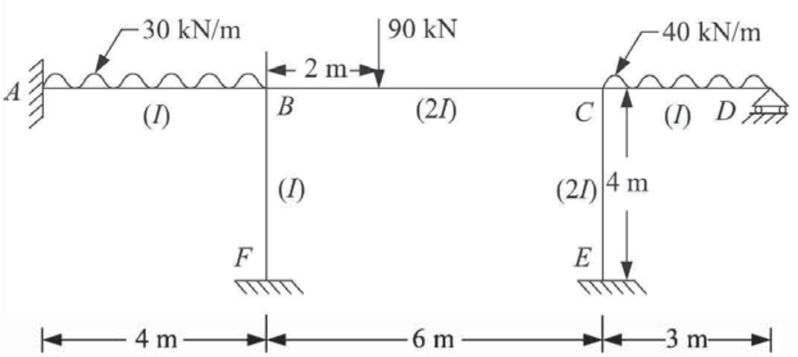
[**Ans:** M_{AB} = +1.77 kNm; M_{BA} = $-M_{BC}$ = 16.90 kNm; M_{CB} = $-M_{CD}$ = 16.89 kNm; M_{DC} = $-M_{DE}$ = 10 kNm.]

3. Analyse the continuous beam shown in figure given below, if the support C settles down by 5 mm. Take Young's modulus = 200 kN/mm^2 and moment of inertia = 3×10^7 mm^4 throughout.



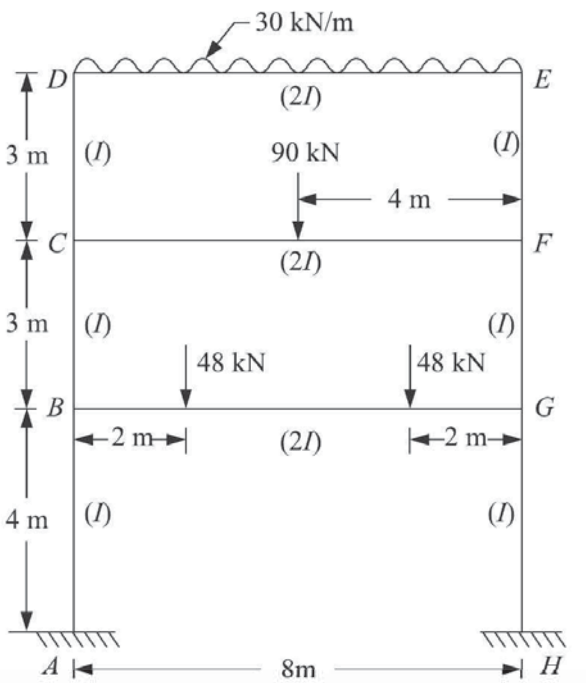
[**Ans:** M_{AB} = 0 kNm; M_{BA} = $-M_{BC}$ = 22.12 kNm; M_{CB} = $-M_{CD}$ = 13.86 kNm; M_{DC} = 0 kNm.]

4. Analyse the frame shown in figure given below by Kani's method.



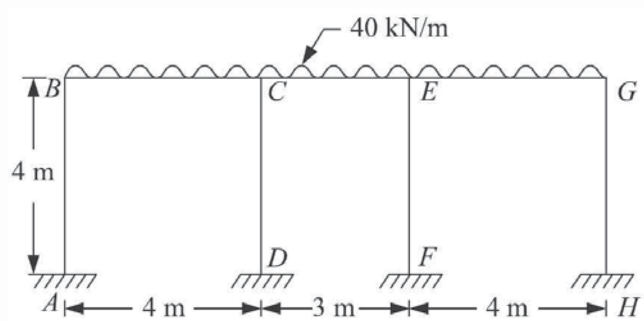
[**Ans:** M_{AB} = −33.91 kNm; M_{BA} = 52.18 kNm; M_{BC} = −64.38 kNm; M_{CB} = 46.86 kNm; M_{CE} = −0.94 kNm; M_{CD} = −45.94 kNm; M_{EC} = −0.47 kNm; M_{FB} = 6.09 kNm; M_{BF} = 12.18 kNm.]

5. Analyse the symmetric frame shown in figure given below.



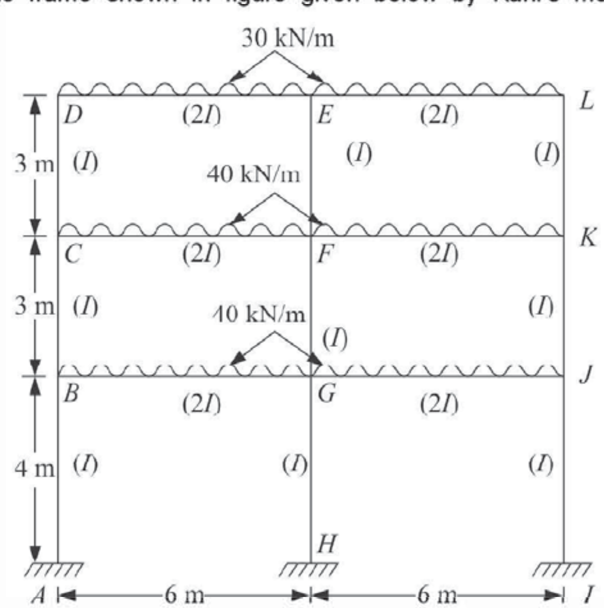
[**Ans:** M_{AB} = –12.04 kNm; M_{BA} = 24.08 kNm;
M_{BG} = –59.96 kNm; M_{BC} = 35.73 kNm;
M_{CD} = 23.22 kNm; M_{CF} = –87.28 kNm;
M_{CD} = 64.03 kNm; M_{DC} = –M_{DE} = –117.34 kNm.]

6. Analyse the symmetric frame shown in figure given below by Kani's method.



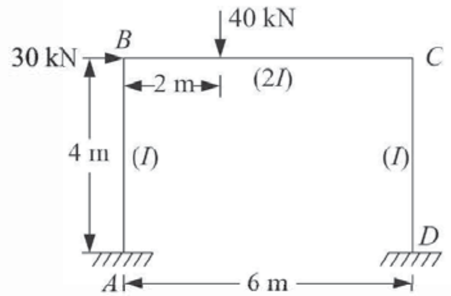
[**Ans:** M_{AB} = 15.13 kNm; M_{BA} = 30.26 kNm; M_{BC} = –30.28 kNm; M_{CB} = 54.26 kNm;
M_{CD} = –14.42 kNm; M_{DC} = –39.62 kNm.]

7. Analyse the symmetric frame shown in figure given below by Kani's method.



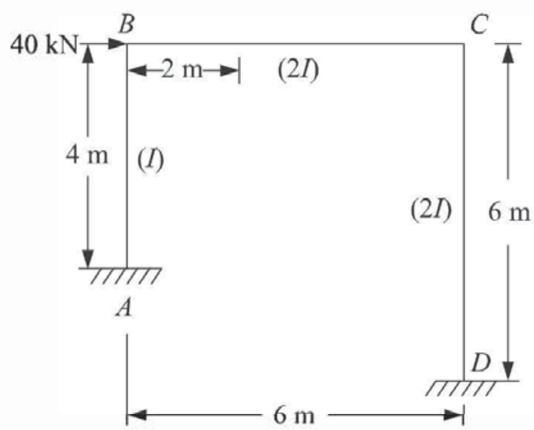
[**Ans:** M_{AB} = 14.51 kNm; M_{BA} = 29.02 kNm;
M_{BG} = –81.32 kNm; M_{BC} = 52.27 kNm;
M_{CB} = 46.52 kNm; M_{CF} = –92.82 kNm;
M_{CD} = 46.28 kNm; M_{DC} = $-M_{DE}$ = 51.79 kNm.]

8. Analyse the frame shown in figure given below by Kani's method.



[**Ans:** M_{AB} = –26.31 kNm; M_{BA} = –9.6 kNm = $-M_{BC}$;
M_{CB} = $-M_{CD}$ = 41.80 kNm; M_{DC} = 42.37 kNm.]

9. Analyse the frame shown in figure given below by Kani's method.



[**Ans:** M_{AB} = –54.73 kNm; M_{BA} = – M_{BC} = –41.4 kNm;
M_{CB} = $-M_{CD}$ = 29.10 kNm; M_{DC} = –34.73 kNm.]

REVIEW QUESTIONS

1. Define the term rotation factor as used in Kani's method and derive the expression for it.
2. Explain the terms:
   (a) Displacement factor
   (b) Storey shear
   (c) Storey moment